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Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences

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0. Introduction

Let Γ be a subgroup of $SL_2(\mathbf{Z})$ of finite index. It is well-known that if Γ is a congruence subgroup, then cusp forms on Γ enjoy many arithmetic properties. For example, the cusp form of weight 12 on $\Gamma = SL_2(\mathbf{Z})$

$$\Delta(z) = q \prod_{n \geqq 1} (1-q^n)^{24} = \sum_{n \geqq 1} \tau(n) \cdot q^n,$$

where $q = \exp 2\pi i z$, has the multiplicative property, first proved by Mordell

$$\tau(np) = \tau(n) \cdot \tau(p) - p^{11} \cdot \tau(n/p) \tag{0.1}$$

and satisfies Ramanujan's conjecture

$$|\tau(p)| \leqq 2p^{11/2} \tag{0.2}$$

as was proved by Deligne ([D1], [D3]).

The proofs of these properties rely on the fact that the numbers $\tau(n)$ are the eigenvalues of the Hecke operators T_n acting on the (one-dimensional) space of cusp forms of weight 12. For subgroups Γ which are not congruence subgroups, the Hecke operators do not exist (as the double cosets $\Gamma \cdot \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \cdot \Gamma \subseteq GL_2(\mathbf{Q})$ are not, in general, finite unions of single cosets). Nevertheless, computations by Atkin and Swinnerton-Dyer [A-SwD] suggest that certain p-adic analogues of (0.1), (0.2) should hold for cusp forms on non-congruence subgroups. For forms of weight 2, their predictions have been confirmed ([A-SwD], [C], [Di], [H], [K2]). In this article we treat the case of forms of arbitrary even weight >2. The methods also apply to forms of odd weight $\geqq 3$, with minor modifications, and to non-cusp forms; see 5.10 below.

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We now state our results in the simplest possible case. Assume that Γ is defined over $\mathbf{Q}$ (in the sense of 5.1 below), and that the space of cusp forms of even weight $w>2$ is one-dimensional. Then a non-zero form may be chosen with Fourier expansion (where μ is the width of the cusp $i\infty$)

$$\sum_{n\geqq 1} a(n)\cdot\exp(2\pi i n z/\mu)$$

in such a way that for a certain $M\geqq 1$ we have

$$k^n\cdot a(n)\in\mathbf{Z}[1/M]$$

where $k\in\mathbf{C}$, $k^\mu\in\mathbf{Z}[1/M]^*$.

Theorem. *For almost all primes p there is an integer A_p, with $|A_p|\leqq 2p^{(w-1)/2}$, such that for all $n\geqq 1$,*

$$\operatorname{ord}_p(a(np)-A_p\cdot a(n)+p^{w-1}\cdot a(n/p))\geqq(\operatorname{ord}_p n+1)(w-1). \tag{0.3}$$

(Since the coefficients $a(n)$ are not rational, some care is needed in the interpretation of the left hand expression; see 5.4 below, and § 5.2 of [A-SwD].)

We also identify the numbers $\{A_p\}$ with the traces of Frobenius elements in a two-dimensional l-adic representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, constructed according to the procedure of [D1], and give some description of the finite set of exceptional primes. For a full statement of results, see 5.2, 5.4 and 5.6 below.

For an example of the above theorem, we may take $\Gamma=\Gamma_{711}$, the unique subgroup of $SL_2(\mathbf{Z})$ of index 9 with a cusp of width 7 at $i\infty$ and inequivalent cusps of width 1 at ± 2. The space of cusp forms of weight $w=4$ on Γ is then one-dimensional; this case is described in [A-SwD]. (It is interesting to note that for this example computations suggest another interpretation for the integers A_p; they appear to be the eigenvalues of $\{T_p\}$ on a certain cusp form of weight 4 on the congruence subgroup Γ_0 (14).)

To illustrate the nature of our proof, we sketch the analogue for forms of weight two (see [K2] for a detailed account of this case, and applications). As Atkin and Swinnerton-Dyer had already observed, the above theorem then amounts to the following:

Let E be an elliptic curve over $\mathbf{Z}_p$, and ω a regular differential on $E/\mathbf{Z}_p$. Let t be a uniformiser along the zero section of E. Then if ω has the expansion

$$\omega=\sum_{n\geqq 1} a(n)\cdot t^{n-1}\cdot dt,$$

the congruences (0.3) *hold, with* $A_p=1+p-|E(\mathbf{F}_p)|$.

We may view ω as an element of the de Rham cohomology

$$H^1_{DR}(E/\mathbf{Z}_p):=\mathbf{H}^1(E,\Omega^{\bullet}_{E/\mathbf{Z}_p}) \tag{0.4}$$

and its power series expansion as an element of the "local" cohomology

$$H^1_{DR}(\hat{E}/\mathbf{Z}_p)=\frac{\mathbf{Z}_p[\![t]\!]\cdot dt}{d(\mathbf{Z}_p[\![t]\!])}. \tag{0.5}$$

According to the principles of crystalline cohomology, acting functorially on each of these groups there is an operator F, whose characteristic polynomial on the first is $T^2 - A_p T + p$, and whose action on the second can be calculated as the action of any lifting of Frobenius - for example, the map $t \mapsto t^p$. Thus

$$\begin{aligned}&(F^2 - A_p \cdot F + p)(\sum a(n) \cdot t^{n-1} \cdot dt)\\ &\quad = \sum a(n)(p^2 t^{np^2-1} - A_p \cdot p t^{np-1} + p t^{n-1}) \cdot dt\\ &\quad \in d(\mathbf{Z}_p[\![t]\!])\end{aligned}$$

from which we recover (0.3) with a power of p missing; to supply it, consider, instead of the de Rham complex, the complex

$$p \cdot \mathcal{O}_E \to \Omega^1_{E/\mathbf{Z}_p} \tag{0.6}$$

whose cohomology also has a crystalline interpretation (as the crystalline cohomology of the sheaf $\mathscr{I}_{E \otimes \mathbf{F}_p/\mathbf{Z}_p}$, see [B-O] 7.23).

A cusp form of weight >2 can be regarded as a differential form on the modular curve, with coefficients in a line bundle. We are therefore led to consider a de Rham cohomology group with non-constant coefficients, analogous to the Eichler-Shimura parabolic cohomology [Sh] and Deligne's l-adic theory [D1]. The system of coefficients is constructed from the cohomology of the universal elliptic curve, and the analogue of the description (0.5) of the "local" cohomology is provided by the theory of the Tate curve, and in particular the explicit calculations of [K1], Appendix.

The reader will observe that, in contrast to the case of weight 2 (when the "axioms" consist only of a curve, a marked point on it, and a uniformising parameter), in the general case the modular properties of the situation are fully exploited (cf. also 2.13.ii below); in particular, the choice of uniformising parameter is critical. On the other hand, the modular curves themselves which occur can be more or less arbitrary; indeed, by the theorem of Belyi [Be], *any* (projective, nonsingular, irreducible) curve over $\bar{\mathbf{Q}}$ can be realised (in many ways) as a quotient $\Gamma \backslash \mathfrak{H}^*$ for some subgroup $\Gamma \leqq SL_2(\mathbf{Z})$ of finite index.

In order to compare the constants A_p with Deligne's l-adic representation, we require a mild generalisation of the Monsky trace formula [M]; this is to be published separately, together with some other facts used here [S].

We now mention some points not raised in the main body of the text.

i) We have assumed throughout that Γ is defined over $\mathbf{Q}$. However this is primarily a notational convenience, and analogous results hold when $\mathbf{Q}$ is replaced by an arbitrary algebraic number field. The interested reader will be able to carry out the necessary modifications.

ii) It is reasonable to suppose that there is a direct connection between the groups $L_k(N, \mathbf{Z}_p)$ of §§2, 3 and the formal groups introduced by Oda [O]. (In fact, Oda's theory was the starting point for the present work).

As should be clear to the reader, we are indebted to the work of Deligne, Dwork, and Katz, amongst others. The author is very grateful to Professors Deligne and Katz for valuable conversations, and to Dr. Birch for drawing the problem to his attention, and for continual encouragement. He wishes to thank the referee, whose suggestions led to a number of improvements to the paper. This work was begun while the author was receiving support from the SERC, and he also benefited from the hospitality of the IHES in 1982; to these organisations he extends his gratitude.

1. Notations and conventions

If X is a scheme, and N is a non-zero integer, we write

$$\begin{aligned} &\Omega^1_X && \text{for } \Omega^1_{X/\mathbf{Z}} \\ &X[1/N] && \text{for } X \underset{\operatorname{Spec}\mathbf{Z}}{\times} \operatorname{Spec}\mathbf{Z}[1/N]. \end{aligned}$$

If X is smooth over S, and $Z \subset X$ is a smooth S-divisor, we write $\Omega^1_{X/S}(\log Z)$ for the (locally free) sheaf of relative differentials with at worst logarithmic poles along Z.

If A is a group (or group scheme), and N is an integer, ${}_NA$ denotes the kernel of multiplication by N on A.

For a locally free sheaf $\mathscr{E}$ on a scheme X, and an integer k, we write

$$\mathscr{E}^k := \begin{cases} \otimes^k \mathscr{E} & \text{if } k>0; \\ \mathcal{O}_X & \text{if } k=0; \quad \text{and} \\ \mathscr{H}om(\mathscr{E}^{-k}, \mathcal{O}_X) & \text{if } k<0. \end{cases}$$

$\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}_p$, $\mathbf{Z}_p$, $\mathbf{F}_q$ have their usual meanings. $\bar{\mathbf{Q}}_p$ denotes an algebraic closure of $\mathbf{Q}_p$, and $\mathbf{Q}_p^{\mathrm{nr}}$ the maximal unramified subfield. The ordinal function on $\bar{\mathbf{Q}}_p$ is normalised with $\operatorname{ord}_p(p)=1$, and

$$\begin{aligned} \bar{\mathbf{Z}}_p &:= \{x \in \bar{\mathbf{Q}}_p : \operatorname{ord}_p(x) \geqq 0\} \\ \mathbf{Z}_p^{\mathrm{nr}} &:= \bar{\mathbf{Z}}_p \cap \mathbf{Q}_p^{\mathrm{nr}}. \end{aligned}$$

If A is a ring (commutative, with 1), A^* denotes the group of units of A.

$\mathfrak{H}$ denotes the Poincaré upper half-plane, and $\mathfrak{H}^*$ the "compactified" half-plane $\mathfrak{H} \cup \mathbf{P}^1(\mathbf{Q})$. The running variable on $\mathfrak{H}$ is denoted τ, except in the introduction, when it is denoted z.

Except in the introduction, we follow the convention of [D1], by which the weight of a modular form is an integer $k+2$. *Unless stated to the contrary, it is always assumed that* $k>0$ (we do not consider here forms of weight 2, in order to simplify the exposition).

$\Gamma(N)$ is the subgroup $\left\{g \in SL_2(\mathbf{Z}) : g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} N)\right\}$ and $\pm\Gamma(N)$ is its image in $PSL_2(\mathbf{Z})$.

In this paper the notion of an action of a group scheme is often employed. To avoid confusion, recall:

if G/S is a group scheme, and X is an S-scheme, an action of G on X is a morphism

$$\pi\colon G \underset{S}{\times} X \to X$$

satisfying certain compatibilities;

if $\mathscr{E}$ is an $\mathcal{O}_X$-module, an action of G on $\mathscr{E}$ is an isomorphism

$$\chi\colon \pi^*\mathscr{E} \xrightarrow{\sim} pr_2^*\mathscr{E},$$

where $pr_2\colon G \underset{S}{\times} X \to X$ is the second projection, subject to certain compatibilities; similarly for a complex of $\mathcal{O}_X$-modules with $\mathcal{O}_S$-linear differential, an étale sheaf on X, etc.;

if $X = S = \operatorname{Spec} W(\mathbf{F}_q)$, and (M, F) is an F-crystal on S (so that $F\colon \sigma^*M \to M$, where σ is the Frobenius endomorphism of S), and G/S is finite étale, an action of G on (M, F) is equivalent to giving, for each $g \in G(W(\bar{\mathbf{F}}_q))$, a $W(\bar{\mathbf{F}}_q)$-linear endomorphism $\tilde{g}$ of $M \otimes W(\bar{\mathbf{F}}_q)$, commuting with the natural σ-linear extension of F, and satisfying various compatibilities.

$=$ denotes equality or canonical isomorphism.

$:=$ denotes that the right hand expression is the definition of the left hand one.

2. Algebraic theory

In 2.1–2.5 we review, mostly without proof, well-known properties of modular curves. For proofs, we refer to [D2], [D–R], and [K1].

2.1. Let N be a positive integer. We denote by $X(N)$ the coarse moduli scheme associated to the functor F_N on $\mathbf{Z}[1/N]$-schemes S:

$$F_N(S) = \left\{\begin{array}{l}\text{isomorphism classes of generalised elliptic curves } E/S,\\ \text{whose geometric fibres are smooth or Néron } N\text{-gons,}\\ \text{together with an isomorphism } \alpha\colon {}_NE \xrightarrow{\sim} \boldsymbol{\mu}_N \times \mathbf{Z}/N \text{ of}\\ \text{determinant one}\end{array}\right\}$$

(cf. [D–R], V.4.4). $X(N)$ is smooth and proper over $\mathbf{Z}[1/N]$, and its geometric fibres are connected curves. The open subscheme $Y(N)$, classifying smooth E/S, is the complement of a closed subscheme $Z(N)$, which is finite and étale over $\mathbf{Z}[1/N]$.

If $N = 1$, the modular invariant j defines isomorphisms ([D–R] VI.1.1)

$$X(1) \xrightarrow{\sim} \mathbf{P}^1_{\mathbf{Z}}, \qquad Y(1) \xrightarrow{\sim} \mathbf{A}^1_{\mathbf{Z}}. \tag{2.11}$$

If $(E/S, \alpha) \in F_N(S)$ is as above, and $g \in SL(\boldsymbol{\mu}_N \times \mathbf{Z}/N)(S)$, then $(E/S, g \circ \alpha) \in F_N(S)$; this defines an action (on the left) of

$$G_N = SL(\boldsymbol{\mu}_N \times \mathbf{Z}/N) \tag{2.1.2}$$

on $X(N)$, which takes $Z(N)$ to itself; the subgroup $\{\pm 1\} \subset G_N$ acts trivially, and

$$G_N \backslash X(N) \simeq X(1)[1/N]. \tag{2.1.3}$$

Let $C/\mathbf{Z}[1/N]$ denote the standard Néron N-gon ([D-R] II.1.1), so that $C^{\text{reg}} = \mathbf{G}_m \times \mathbf{Z}/N$. The canonical isomorphism ([D-R] II.1.18)

$$_N C \xrightarrow{\sim} \boldsymbol{\mu}_N \times \mathbf{Z}/N$$

defines a section $\underline{\infty}$ of $X(N)$ over $\mathbf{Z}[1/N]$. The stabiliser of $\underline{\infty}$ in G_N is the subgroup scheme

$$\pm U_N := \left\{ \pm \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \operatorname{Hom}(\mathbf{Z}/N, \boldsymbol{\mu}_N) \right\}$$

whence

$$Z(N) \simeq G_N / \pm U_N$$

(cf. [D-R] VI.5.1). In the sequel we shall generally consider behaviour only at the component $\underline{\infty}$ of $Z(N)$, the other components being obtainable from $\underline{\infty}$ by translation by G_N.

2.2. Let $\operatorname{Tate}(q)/\mathbf{Z}[\![q^{1/N}]\!]$ denote the Tate curve with N sides ([D-R] VII.1.16). The canonical level N structure on Tate(q) ([D-R] VII 1.16.4) defines a morphism

$$\psi \colon \operatorname{Spec} \mathbf{Z}[1/N][\![q^{1/N}]\!] \to X(N) \tag{2.2.1}$$

identifying $R_N = \mathbf{Z}[1/N][\![q^{1/N}]\!]$ with the formal completion of $X(N)$ along $\underline{\infty}$. Denote by ψ_0 the induced morphism

$$\psi_0 \colon \operatorname{Spec} R_N[q^{-1}] \to Y(N). \tag{2.2.2}$$

The complex manifold $X(N)(\mathbf{C})$ is isomorphic to the quotient space $\Gamma(N)\backslash\mathfrak{H}^*$; the $\Gamma(N)$-equivalence classes of cusps correspond to $Z(N)(\mathbf{C})$, and the cusp $i\infty$ to the point $\underline{\infty}_{\mathbf{C}}$. The morphism ψ then identifies $q^{1/N}$ with the local parameter $\exp(2\pi i\tau/N)$ on $\Gamma(N)\backslash\mathfrak{H}^*$ at $i\infty$.

2.3. Assume from now on that $N \geqq 3$. Then $X(N)$ represents the functor F_N; there is a "universal generalised elliptic curve"

$$\begin{tikzcd} E_{\text{univ}} \arrow[r, "f"] & X(N) \arrow[l, bend left, "e"] \end{tikzcd}$$

with zero section e. The group G_N acts on E_{univ} over its action on $X(N)$.

We have the invertible sheaf

$$\omega = e^* \Omega^1_{E_{\text{univ}}/X(N)} = f_* \Omega^{\text{reg}}_{E_{\text{univ}}/X(N)}$$

([D-R] II.1.6; [D2] §1) and a canonical generator ω for $\psi^*\omega$ (denoted dx/x in [D-R] VII.1.16.2, and ω_{can} in [K1] Appendix 1).

Let us provisionally write $\mathscr{E}$ for the relative de Rham cohomology of the restriction of E_{univ} to $Y(N)$:

$$\mathscr{E} = \mathbf{R}^1 f_* \Omega^{\boldsymbol{\cdot}}_{f^{-1}(Y(N))/Y(N)}.$$

The exact sequence (the Hodge filtration)

$$0 \to \omega|_{Y(N)} \to \mathscr{E} \to \omega^{-1}|_{Y(N)} \to 0 \tag{2.3.1}$$

is auto-dual with respect to the cup-product on $\mathscr{E}$, and we have the Gauss-Manin connection

$$\nabla\colon \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{Y(N)}. \tag{2.3.2}$$

G_N acts on ω and $\mathscr{E}$, compatibly with (2.3.1) and (2.3.2).

2.4. By [K1], A.1.3 *et seq.*, we have

$$\psi_0^* \mathscr{E} = R_N[q^{-1}] \cdot \omega \oplus R_N[q^{-1}] \cdot \xi \tag{2.4.1}$$

where

$$\xi = \nabla \left(q \frac{d}{dq} \right) (\omega), \quad \text{and } \nabla \xi = 0.$$

We may therefore specify an extension of $\mathscr{E}$ to a G_N-sheaf on $X(N)$, which we now call $\mathscr{E}$, by stipulating at $\underline{\infty}$ that $\{\omega, \xi\}$ extends to a basis of $\psi^* \mathscr{E}$.

Since $\langle \omega, \xi \rangle = 1$, we have an exact sequence of $\mathscr{O}_{X(N)}$-modules

$$0 \to \omega \to \mathscr{E} \to \omega^{-1} \to 0 \tag{2.4.2}$$

extending (2.3.1), and (2.3.2) extends to a connection with logarithmic poles

$$\nabla\colon \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{X(N)}(\log Z(N)). \tag{2.4.3}$$

This gives rise to an isomorphism

$$\omega^2 \xrightarrow{\sim} \Omega^1_{X(N)}(\log Z(N))$$

by forming the composite

$$\theta\colon \omega \hookrightarrow \mathscr{E} \xrightarrow{\nabla} \mathscr{E} \otimes \Omega^1_{X(N)}(\log Z(N)) \twoheadrightarrow \omega^{-1} \otimes \Omega^1_{X(N)}(\log Z(N)). \tag{2.4.4}$$

([K1] A.1.3.17; [D-R] VI.4.5.2.)

2.5. If A is a $\mathbf{Z}[1/N]$-algebra (or even module), the module of cusp forms of level N and weight $k+2 \in \mathbf{Z}$ is

$$S_{k+2}(N, A) := H^0(X(N),\ \omega^k \otimes \Omega^1_{X(N)} \otimes A).$$

The q-expansion map

$$\begin{tikzcd} S_{k+2}(N, A) \arrow[r] & q^{1/N} R_N \otimes A \\ f \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \tilde{f} \arrow[u, phantom, sloped, "\in"] \end{tikzcd} \tag{2.5.1}$$

is defined by

$$\psi^*(f) = \tilde{f} \cdot \omega^k \cdot \frac{dq}{q} \in \psi^*(\omega^k \otimes \Omega^1_{X(N)} \otimes A).$$

The q-expansion principle ([D-R] VII.3.9; [K1] 1.6.1) affirms that this map is injective, and that if $A \subseteq A'$, and $f \in S_{k+2}(N, A')$, then f belongs to $S_{k+2}(N, A)$ if and only if the coefficients of $\tilde{f}$ are in A. Standard base-changing techniques show that for every $k \geqq 0$ the modules $S_{k+2}(N, A)$ are free, and

$$S_{k+2}(N, A) = S_{k+2}(N, \mathbf{Z}[1/N]) \otimes A. \tag{2.5.2}$$

2.6. Assume from now on that k is strictly positive. We define a complex of sheaves $\Omega^{\bullet}(\mathscr{E}_k)$ on $X(N)$ as follows:

$$\begin{aligned} \Omega^0(\mathscr{E}_k) &:= \mathscr{E}_k := \operatorname{Sym}^k \mathscr{E} \\ \Omega^1(\mathscr{E}_k) &:= \nabla_k(\mathscr{E}_k) + \mathscr{E}_k \otimes \Omega^1_{X(N)} \\ &\subseteq \mathscr{E}_k \otimes \Omega^1_{X(N)}(\log Z(N)) \end{aligned} \tag{2.6.1}$$

where

$$\nabla_k \colon \mathscr{E}_k \to \mathscr{E}_k \otimes \Omega^1_{X(N)}(\log Z(N))$$

is the k^{th} symmetric power of ∇. Thus

$$\Omega^{\bullet}(\mathscr{E}_k)_{|Y(N)} = \mathscr{E}_{k|Y(N)} \otimes \Omega^{\bullet}_{Y(N)}$$

and

$$\begin{aligned} \psi^* \Omega^0(\mathscr{E}_k) &= \bigoplus_{r=0}^{k} \omega^{k-r} \cdot \xi^r \cdot R_N \\ \psi^* \Omega^1(\mathscr{E}_k) &= q^{1/N} \cdot \omega^k \cdot \frac{dq}{q} \cdot R_N \oplus \bigoplus_{r=1}^{k} \omega^{k-r} \cdot \xi^r \cdot \frac{dq}{q} \cdot R_N \end{aligned} \tag{2.6.2}$$

with

$$\nabla_k(\omega^{k-r} \cdot \xi^r) = (k-r)\omega^{k-r-1} \cdot \xi^{r+1} \cdot \frac{dq}{q}.$$

The sheaves $\Omega^i(\mathscr{E}_k)$ are locally free, and $\Omega^{\bullet}(\mathscr{E}_k)$ is the smallest subcomplex of $\mathcal{O}_{X(N)}$-modules and differential operators of $\mathscr{E}_k \otimes \Omega^{\bullet}_{X(N)}(\log Z(N))$ which contains $\mathscr{E}_k \otimes \Omega^i_{X(N)}$ in degree i, for $i = 0, 1$.

Define

$$\begin{aligned} L_k(N, A) &:= \mathbf{H}^1(X(N), \Omega^{\bullet}(\mathscr{E}_k) \otimes A) \\ L_k^{\infty}(N, A) &:= H^1(\psi_A^* \Omega^{\bullet}(\mathscr{E}_k)) \end{aligned} \tag{2.6.3}$$

where

$$\psi_A \colon \operatorname{Spec} A[\![q^{1/N}]\!] \to X(N)$$

is the natural extension of ψ.

By construction, $\Omega^{\bullet}(\mathscr{E}_k)$ is a G_N-stable subcomplex of $\mathscr{E}_k \otimes \Omega^{\bullet}_{X(N)}(\log Z(N))$, whence G_N acts on $L_k(N, A)$.

2.7. **Theorem.** *Let A be a $\mathbf{Z}[1/N]$-algebra in which $k!$ is invertible.*

i) *There is an exact sequence of A-modules*

$$0 \to S_{k+2}(N, A) \to L_k(N, A) \to S_{k+2}(N, A)^{\vee} \to 0$$

(where ${}^{\vee}$ denotes A-dual), compatible with the action of G_N.

ii) *There is an isomorphism*

$$L_k^{\infty}(N, A) \xrightarrow{\sim} \operatorname{coker}\left(\left(q\frac{d}{dq}\right)^{k+1} : q^{1/N} A[\![q^{1/N}]\!] \longrightarrow q^{1/N} A[\![q^{1/N}]\!]\right).$$

iii) i) *and* ii) *are compatible with base-change* $A \to A'$.

iv) *If* $f \in S_{k+2}(N, A)$, *then its image in* $L_k^\infty(N, A)$ *is represented, in terms of the isomorphism* ii), *by the class of its q-expansion* $\tilde{f}$.

Proof. The Hodge filtration (2.4.2) on $\mathscr{E}$ defines a decreasing filtration

$$\mathscr{E}_k = F^0_{\mathrm{Hdg}} \supseteq F^1_{\mathrm{Hdg}} \supseteq \ldots \supseteq F^{k+1}_{\mathrm{Hdg}} = 0 \tag{2.7.1}$$

on $\mathscr{E}_k$, with associated graded parts

$$\mathrm{Gr}^i_{\mathrm{Hdg}}(\mathscr{E}_k) = F^i_{\mathrm{Hdg}}/F^{i+1}_{\mathrm{Hdg}} \simeq \omega^{(2i-k)}, \qquad 0 \leqq i \leqq k$$

such that

$$V_k(F^i_{\mathrm{Hdg}}) \subseteq F^{i+1}_{\mathrm{Hdg}} \otimes \Omega^1_{X(N)}(\log Z(N)), \tag{2.7.2}$$

a simple case of Griffiths transversality. We can accordingly define a filtration $F^\cdot$ on the complex $\Omega^\cdot(\mathscr{E}_k)$ by

$$\begin{aligned} F^i(\Omega^0(\mathscr{E}_k)) &:= F^i_{\mathrm{Hdg}}(\mathscr{E}_k) \\ F^i(\Omega^1(\mathscr{E}_k)) &:= \Omega^1(\mathscr{E}_k) \cap F^{i-1}_{\mathrm{Hdg}}(\mathscr{E}_k) \otimes \Omega^1_{X(N)}(\log Z(N)) \end{aligned} \tag{2.7.3}$$

which has

$$\begin{aligned} \mathrm{Gr}^0_F(\Omega^\cdot(\mathscr{E}_k)) &= [\omega^{-k} \longrightarrow 0] \\ \mathrm{Gr}^i_F(\Omega^\cdot(\mathscr{E}_k)) &= [\omega^{2i-k} \xrightarrow{\delta_i} \omega^{2i-k-2} \otimes \Omega^1_{X(N)}(\log Z(N))] \\ \mathrm{Gr}^{k+1}_F(\Omega^\cdot(\mathscr{E}_k)) &= [0 \longrightarrow \omega^k \otimes \Omega^1_{X(N)}]. \end{aligned}$$

This follows easily from the description (2.6.2) of the complex $\Omega^\cdot(\mathscr{E}_k)$. Here

$$\delta_i = i \cdot (\theta \otimes \mathrm{id}_{\omega^{2i-k-1}}), \qquad 1 \leqq i \leqq k. \tag{2.7.4}$$

In fact, if

$$\alpha^i \cdot \lambda \in F^i_{\mathrm{Hdg}}(\mathscr{E}_k), \qquad \alpha \in \omega, \qquad \lambda \in \mathscr{E}_{k-i}$$

(sections over some open subset of $X(N)$) we have

$$\begin{aligned} V_k(\alpha^i \cdot \lambda) &= i\alpha^{i-1} \cdot \lambda \cdot V\alpha + \alpha^i \cdot V_{k-i}(\lambda) \\ &\equiv i\alpha^{i-1} \cdot \lambda \cdot V\alpha \qquad (\mathrm{mod}\, F^i_{\mathrm{Hdg}}) \end{aligned}$$

and so (2.7.4) is an immediate consequence of the definition (2.4.4) of θ. In particular, the complexes

$$\mathrm{Gr}^i_F(\Omega^\cdot(\mathscr{E}_k)) \otimes A, \qquad 1 \leqq i \leqq k,$$

are acyclic, since θ is an isomorphism and $k!$ is invertible in A. Since the sheaves $\mathrm{Gr}^i_F(\Omega^j(\mathscr{E}_k))$ are locally free,

$$\mathrm{Gr}^i_F(\Omega^j(\mathscr{E}_k)) \otimes A \simeq \mathrm{Gr}^i_F(\Omega^j(\mathscr{E}_k) \otimes A).$$

Applying the spectral sequence for the derived functors of $\mathbf{H}^0$ and the filtered complex $(\Omega^\cdot(\mathscr{E}_k) \otimes A, F)$, we then get a long exact sequence:

$$\begin{aligned}
0&\to \mathbf{H}^0(X(N),\ \Omega^{\cdot}(\mathscr{E}_k)\otimes A)\to H^0(X(N),\ \omega^{-k}\otimes A)\\
&\to H^0(X(N),\ \omega^k\otimes\Omega^1_{X(N)}\otimes A)\to L_k(N,A)\to H^1(X(N),\ \omega^{-k}\otimes A)\\
&\to H^1(X(N),\ \omega^k\otimes\Omega^1_{X(N)}\otimes A)\to \mathbf{H}^2(X(N),\ \Omega^{\cdot}(\mathscr{E}_k))\to 0.
\end{aligned}$$

Now $H^1(X(N),\ \omega^{-k}\otimes A)\simeq S_{k+2}(N,A)^{\vee}$ is free, and ω is ample, by [D-R] VII.3.4, and so $H^0(X(N),\ \omega^{-k}\otimes A)$ and its dual $H^1(X(N),\ \omega^k\otimes\Omega^1_{X(N)}\otimes A)$ vanish, giving i).

Now consider the spectral sequence for the cohomology of the filtered complex $(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k), F)$. It gives an exact sequence

$$\begin{aligned}
0&\longrightarrow H^0(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))\longrightarrow \psi_A^*(\omega^{-k})\\
&\xrightarrow{\mathscr{D}} \psi_A^*(\omega^k\otimes\Omega^1_{X(N)})\longrightarrow L_k(N,A)\longrightarrow 0.
\end{aligned}$$

Let us calculate the homomorphism $\mathscr{D}$ in terms of the basis $\{\omega,\xi\}$ of $\psi_A^*(\mathscr{E})$. Unravelling the spectral sequence, one sees that $\mathscr{D}$ fits into a commutative diagram

$$\begin{tikzcd}
\psi_A^*(\omega^{-k})=H^0(\mathrm{Gr}_F^0(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))) \arrow[r, "\delta"] \arrow[d, "\mathscr{D}"'] & H^1(F^1(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k)) \\
\psi^*(\omega^k\otimes\Omega^1_{X(N)}) \arrow[r, phantom, "="] & H^1(F^{k+1}(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k)) \arrow[u, "\wr"', "\varepsilon"]
\end{tikzcd}$$

where δ is the connecting homomorphism in the long exact sequence of cohomology for

$$0\to F^1(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))\to\psi_A^*\Omega^{\cdot}(\mathscr{E}_k)\to\mathrm{Gr}_F^0(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))\to 0$$

and ε is induced by the quasi-isomorphism

$$F^{k+1}(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))\to F^1(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k)).$$

Let

$$a\cdot\xi^k\in\psi_A^*\omega^{-k},\qquad a\in A[\![q^{1/N}]\!].$$

Then $\delta(a\cdot\xi^k)$ is the class in $H^1(F^1(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k)))$ of

$$\nabla_k(a\cdot\xi^k)=q\frac{da}{dq}\cdot\xi^k\cdot\frac{dq}{q}.$$

Let

$$\zeta=\sum_{i=1}^k\frac{1}{i!}\left(-q\frac{d}{dq}\right)^i(a)\cdot\omega^i\cdot\xi^{k-i}\in F^1(\psi_A^*\mathscr{E}_k).$$

Then

$$\nabla_k(\zeta)+\nabla_k(a\cdot\xi^k)=\frac{(-1)^k}{k!}\left(q\frac{d}{dq}\right)^{k+1}(a)\cdot\omega^k\cdot\frac{dq}{q}$$

is an element of $F^{k+1}(\psi_A^*\Omega^{\cdot}(\mathscr{E}_k))$, and thus

$$\mathscr{D}(a\cdot\xi^k)=\frac{(-1)^k}{k!}\left(q\frac{d}{dq}\right)^{k+1}(a)\cdot\omega^k\cdot\frac{dq}{q}.$$

This proves ii); the compatibility iii) is obvious from the definitions, and iv) follows from the proof of ii) and the definition (2.5.1) of the q-expansion map.

2.8. *Remarks.* i) $L_k(N, A)$ is thus an algebraic de Rham analogue of the parabolic cohomology ([D1], [Sh]), the exact sequence i) being its "Hodge filtration". It has many expected properties (duality, comparison theorem when $A=\mathbf{C}$, Hecke operators, ...).

ii) As is clear from the proof, the hypothesis that $k!$ is invertible in A cannot be removed. However, there is another candidate for L_k for which the theorem holds without this hypothesis, at any rate if A is $\mathbf{Z}$-flat. To define it, consider the $\mathcal{O}_{X(N)}$-module $\Gamma^k(\mathscr{E}; \omega)$, which is the submodule of $\mathscr{E}_k \otimes \mathbf{Q}$ spanned by local sections of the form

$$\frac{x_1^{a_1} \dots x_r^{a_r}}{a_1! \dots a_r!} \cdot y_1^{b_1} \dots y_s^{b_s}$$

where $x_i \in \omega$, $y_j \in \mathscr{E}$, and $\sum_{i=1}^{r} a_i + \sum_{j=1}^{s} b_j = k$. Note that $\Gamma^k(\mathscr{E}; \omega)$ is auto-dual (unlike $\mathscr{E}_k$).

If we form a complex $\Omega^{\cdot}(\Gamma^k(\mathscr{E}; \omega))$ in a manner analogous to $\Omega^{\cdot}(\mathscr{E}_k)$, then its $\mathbf{H}^1$ will have the desired property i) of the theorem.

2.9. We require a refinement of the preceding theorem, which we shall apply in a slightly more general setting. We assume that we are given:

i) a normal, connected scheme X, together with a finite flat morphism

$$h\colon X \to X(N)[1/M]$$

for some integers M, N with $N \geqq 3$, $N | M$, such that h is étale over $Y(N)$;

ii) a section $\underline{\infty}'$: $\operatorname{Spec} \mathbf{Z}[1/M] \to X$, such that $h(\underline{\infty}') = \underline{\infty}$;

iii) an action of a closed flat subgroup scheme $G \subseteq G_N[1/M]$ on X, compatible with the action of G_N on $X(N)$.

In what follows we shall simply write $X(N)$ for $X(N)[1/M]$, and likewise for similar expressions.

By Abhyankar's lemma ([SGA1] Exp. XIII §5) X is smooth over $\mathbf{Z}[1/M]$, and $Y = h^{-1}(Y(N))$ is the complement in X of a closed subscheme Z, finite and étale over $\mathbf{Z}[1/M]$; the morphism h is tamely ramified along Z, and the completion of $\mathcal{O}_X$ along $\underline{\infty}'$ is of the form

$$R' = \mathbf{Z}[1/M][[t]], \qquad D \cdot t^{\nu} = q^{1/N}, \tag{2.9.1}$$

for some $D \in \mathbf{Z}[1/M]^*$, and some $\nu \geqq 1$, invertible in $\mathbf{Z}[1/M]$.

Denote by ψ' the associated morphism $\operatorname{Spec} R' \to X$, and by ψ'_A the resultant morphism $\operatorname{Spec} A[[t]] \to X$, for any $\mathbf{Z}[1/M]$-algebra A.

More generally, the completion of $\mathcal{O}_X$ along any component of Z is of the form $B[[u]]$ for an étale integral $\mathbf{Z}[1/M]$-algebra B, where u satisfies

$$\kappa \cdot u^{\nu'} = q^{1/N}, \qquad \kappa \in B^*, \ \nu' \geqq 1,$$

with ν' invertible in $\mathbf{Z}[1/M]$.

Define

$$S_{k+2}(X,A) := H^0(X, h^*\omega^k \otimes \Omega^1_X \otimes A)$$

for an integer $k>0$ and a $\mathbf{Z}[1/M]$-algebra A. Then ψ' gives rise to the "t-expansion map"

$$\begin{tikzcd} S_{k+2}(X,A) \arrow[r] & t\cdot R'\otimes A \\ f \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[r] & \tilde{f} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

such that

$$\psi'^*(f) = f\cdot\omega^k\cdot\frac{dq}{q} = \nu N\cdot f\cdot\omega^k\cdot\frac{dt}{t}.$$

The proof of the q-expansion principle of § 2.5 applies in this case, showing that the t-expansion map is injective, and that if $A\subseteq B$, then $S_{k+2}(X,A) = S_{k+2}(X,B)\cap t\cdot R'\otimes A$; furthermore, the formation of $S_{k+2}(X,A)$ is compatible with base-change.

2.10. Since h is étale away from Z, and tamely ramified along it, we have

$$h^*\Omega^1_{X(N)}(\log Z(N)) = \Omega^1_X(\log Z). \tag{2.10.1}$$

(2.4.3) gives a connection with logarithmic poles

$$\nabla\colon\ h^*\mathscr{E}\to h^*\mathscr{E}\otimes\Omega^1_X(\log Z)$$

with symmetric powers ∇_k.

Define, for $m\geq 1$,

$$\mathscr{E}_{1,m} := \beta^{-1}(m\cdot\omega^{-1})$$
$$\mathscr{E}_{k,m} := \mathrm{Sym}^k(\mathscr{E}_{1,m})$$

where β is the surjection $\mathscr{E}\twoheadrightarrow\omega^{-1}$. On X, we have

$$\nabla_k(m\cdot h^*\mathscr{E}_{k,m}) \subseteq h^*\mathscr{E}_{k,m}\otimes\Omega^1_X(\log Z)$$

by virtue of the transversality (2.7.2). We may therefore define a complex $\mathscr{Q}^{\bullet}_{k,m}$ of sheaves on X by

$$\mathscr{Q}^0_{k,m} := m\cdot h^*\mathscr{E}_{k,m}$$
$$\mathscr{Q}^1_{k,m} := \nabla_k(m\cdot h^*\mathscr{E}_{k,m}) + h^*\mathscr{E}_{k,m}\otimes\Omega^1_X.$$

Thus $\mathscr{Q}^{\bullet}_{k,m}$ is a subcomplex of $h^*\Omega^{\bullet}(\mathscr{E}_k)$, and

$$\mathscr{Q}^{\bullet}_{k,m|Y} = [m\cdot h^*\mathscr{E}_{k,m|Y}\to h^*\mathscr{E}_{k,m|Y}\otimes\Omega^1_Y]$$

$$\psi'^*\,\mathscr{Q}^i_{k,m} = \begin{cases} \displaystyle\bigoplus_{r=0}^{k} m^{r+1}\cdot\omega^{k-r}\cdot\xi^r\cdot R' & (i=0) \\ \displaystyle\omega^k\cdot t\frac{dt}{t}\cdot R'\oplus\bigoplus_{r=1}^{k} m^r\cdot\omega^{k-r}\cdot\xi^r\cdot\frac{dt}{t}\cdot R' & (i=1) \end{cases}$$

with

$$\nabla_k(\omega^{k-r}\cdot\xi^r)=(k-r)\cdot\omega^{k-r-1}\cdot\xi^{r+1}\cdot\nu N\frac{dt}{t}\qquad(0\leqq r<k)$$
$$\nabla_k(\xi^k)=0.$$

The restriction of $\mathcal{Q}^\bullet_{k,m}$ to the formal completion of X along any other component of Z admits a similar description, in view of the remarks in the previous section, and those at the end of § 2.1.

(We shall use later the complex $\mathcal{Q}^\bullet_{k,p}\otimes\mathbf{Z}_p$, for a prime p; it is the appropriate generalisation of the complex (0.6), and provides an extra factor of p^{k+1} in the modulus of the Atkin-Swinnerton-Dyer congruences.)

2.11. Let us examine the filtrations (2.7.1) and (2.7.3) applied to the complex $\mathcal{Q}^\bullet_{k,m}$. We clearly have

$$\begin{aligned}\mathrm{Gr}^i_{\mathrm{Hdg}}(h^*\mathscr{E}_{k,m})&=h^*\,\mathrm{Gr}^i_{\mathrm{Hdg}}(\mathscr{E}_{k,m})\\&=m^{k-i}\cdot h^*\omega^{2i-k}\qquad(0\leqq i\leqq k)\end{aligned}$$

and thus

$$\mathrm{Gr}^0_F(\mathcal{Q}^\bullet_{k,m})=[m^{k+1}\cdot h^*\omega^k\to 0]$$
$$\mathrm{Gr}^i_F(\mathcal{Q}^\bullet_{k,m})=[m^{k-i+1}\cdot h^*\omega^{k-2i}\xrightarrow[\varepsilon_i]{}m^{k-i+1}\cdot h^*\omega^{k-2i-2}\otimes\Omega^1_X(\log Z)]\qquad(1\leqq i\leqq k)$$
$$\mathrm{Gr}^{k+1}_F(\mathcal{Q}^\bullet_{k,m})=[0\to h^*\omega^k\otimes\Omega^1_X]$$

where ε_i is calculated as in (2.7.4). By (2.10.1), $\varepsilon_i=h^*\delta_i$, and as in the proof of Theorem 2.7 the complexes

$$\mathrm{Gr}^i_F(\mathcal{Q}^\bullet_{k,m})\otimes\mathbf{Z}\left[\frac{1}{k!}\right]\qquad(2.11.1)$$

are acyclic, for $1\leqq i\leqq k$. Moreover, if m is not a zero-divisor in A, the construction above commutes with the base-change $\mathbf{Z}[1/M]\to A$. We may therefore generalise the theorem as follows. Define

$${}^{(m)}L_k(X,A):=\mathbf{H}^1(X,\mathcal{Q}^\bullet_{k,m}\otimes A)$$
$${}^{(m)}L_k^\infty(X,A):=H^1(\psi_A'^*\,\mathcal{Q}^\bullet_{k,m});$$

there are natural transitive maps

$${}^{(m)}L_k(X,A)\to{}^{(n)}L_k(X,A)\text{ and }{}^{(m)}L_k^\infty(X,A)\to{}^{(n)}L_k^\infty(X,A)$$

if $n|m$.

Write ∂ for the derivation $q\dfrac{d}{dq}=\dfrac{1}{N\nu}\cdot t\dfrac{d}{dt}$ of $A[[t]]$.

2.12. **Theorem.** *Let A be a $\mathbf{Z}\left[\dfrac{1}{M\cdot k!}\right]$-algebra, and let m be a positive integer which is not a zero-divisor in A.*

i) *There is an exact sequence*

$$0\to S_{k+2}(X,A)\to{}^{(m)}L_k(X,A)\to m^{k+1}\cdot S_{k+2}(X,A)^\vee\to 0$$

compatible with the action of G.

ii) *There is an isomorphism*

$$^{(m)}L_k^{\infty}(X, A) \xrightarrow{\sim} \operatorname{coker}((m\,\partial)^{k+1}\colon\ t\cdot A[\![t]\!] \to t\cdot A[\![t]\!]).$$

iii) i) *and* ii) *are compatible with base-change $A \to A'$, and with the natural transitive maps if $n|m$.*

iv) *If $f \in S_{k+2}(X, A)$, then its image in $^{(m)}L_k^{\infty}(X, A)$ is represented by the class of its t-expansion $\tilde{f}$.*

2.13. *Remarks.* i) If $X = X(N)$, $m = 1$, and $M = N$, then $^{(m)}L_k(X, A)$ reduces to $L_k(N, A)$.

ii) The hypothesis that $h_{|Y}$ is étale is essential; if it is not satisfied, the complexes (2.11.1) will have non-trivial H^1, and so the filtration 2.12.i) on L_k will have other graded parts than S_{k+2} and $S_{k+2}{}^{\vee}$.

iii) Write

$$T_k(X, A) = \mathbf{H}^1(X, h^* \mathscr{E}_k \otimes \Omega_X^{\bullet}(\log Z) \otimes A)$$

and define similarly $^{(m)}T_k(X, A)$, $^{(m)}T_k^{\infty}(X, A)$. Under the hypotheses of 2.12 there is then an exact sequence

$$0 \to M_{k+2}(X, A) \to T_k(X, A) \to S_{k+2}(X, A)^{\vee} \to 0 \qquad (2.13.1)$$

where

$$M_{k+2}(X, A) := H^0(X, \omega^{k+2} \otimes A)$$

is the module of holomorphic modular forms. There is a natural inclusion of the sequence 2.12.i) in this sequence. If $X = X(N)$ and A is a $\mathbf{Q}$-algebra, this inclusion is split by the Hecke algebra.

For general X, the theory of Eisenstein series provides a splitting if $A = \mathbf{C}$; see also 5.10.ii) below for a p-adic splitting.

3. p-adic theory

3.1. Let $N \geqq 3$, and let p be a prime with $(p, 2N) = 1$. Let $\hat{}$ denote p-adic completion, and write $\hat{\psi}$, $\hat{\psi}_0$ for the uniformisations along $\underline{\infty}$

$$\begin{tikzcd}
\operatorname{Spf} \mathbf{Z}_p[\![q^{1/N}]\!] \arrow[r, "\hat{\psi}"] & \widehat{X(N)}. \\
\cup & \\
\operatorname{Spf} \mathbf{Z}_p\widehat{((q^{1/N}))} \arrow[uur, "\hat{\psi}_0"'] &
\end{tikzcd}$$

For the notion of an F-crystal with logarithmic singularities, see [S].

3.2. **Proposition.** *$(\hat{\mathscr{E}}, \nabla)$ is the underlying differential equation of an F-crystal on $\widehat{X(N)}$ with logarithmic singularities along $\widehat{Z(N)}$, whose restriction to $\widehat{Y(N)}$ is the crystalline H^1 of the family*

$$E_{\mathrm{univ}|Y(N)} \otimes \mathbf{F}_p \to Y(N) \otimes \mathbf{F}_p.$$

Moreover, the F-crystal structure on $\hat{\psi}^*\hat{\mathscr{E}}$ *is given by*

$$F_\phi(\omega)=p\cdot\omega;\qquad F_\phi(\xi)=\xi \tag{3.2.1}$$

where ϕ *is the endomorphism of* $\operatorname{Spf}\mathbf{Z}_p[\![q^{1/N}]\!]$ *given by*

$$\phi^*(q^{1/N})=q^{p/N}.$$

Proof. For $(\hat{\mathscr{E}}, \nabla)$ to define an F-crystal with logarithmic singularities, the following data are required: for each open $U\subseteq\widehat{X(N)}$, and each lifting ϕ: $U\to U$ of the absolute Frobenius of $U\otimes\mathbf{F}_p$, satisfying

$$\phi^*(\mathscr{I})\subseteq\mathscr{I}^p \tag{3.2.2}$$

(where $\mathscr{I}$ is the ideal sheaf of $\widehat{Z(N)}$ in U), there is a horizontal homomorphism

$$F_\phi\colon\ \phi^*\hat{\mathscr{E}}_{|U}\to\hat{\mathscr{E}}_{|U}$$

which becomes an isomorphism when tensored with $\mathbf{Q}$, satisfying certain compatibilities.

The F-crystal structure of $\hat{\mathscr{E}}_{|\widehat{Y(N)}}$ is a consequence of the main comparison theorem of crystalline cohomology ([B–O] Ch. VII). As explained in § 1.6 of [*S*], it then suffices to find one neighbourhood U of $\widehat{Z(N)}$ in $\widehat{X(N)}$, and one lifting ϕ of Frobenius on U satisfying (3.2.2), for which the mapping

$$F_\phi\colon\ \phi^*\hat{\mathscr{E}}_{|U\cap\widehat{Y(N)}}\to\hat{\mathscr{E}}_{|U\cap\widehat{Y(N)}}$$

(given by the F-crystal structure of $\hat{\mathscr{E}}_{|\widehat{Y(N)}}$) extends to U.

For this, let U be the complement of the supersingular points of $\widehat{X(N)}$, and recall the definition of the canonical lifting ϕ of Frobenius to U ([K1] Ch. 3; the "Deligne-Tate mapping" in the terminology of [Dw1], [Dw2]). There is a finite flat subgroup scheme (the canonical subgroup)

$$H\subset{}_pE_{\mathrm{univ}|U}$$

whose reduction modulo p is the kernel of Frobenius. If E' denotes the quotient $E_{\mathrm{univ}|U}$ by H, then there is a level N structure α' on E' given by the commutative diagram

$$\begin{tikzcd}
E_{\mathrm{univ}|U} \arrow[r] & E' \\
{}_NE_{\mathrm{univ}|U} \arrow[u, hook] \arrow[r, "\sim"] \arrow[d, "\simeq"', "\alpha"] & {}_NE' \arrow[u, hook] \arrow[d, "\simeq"', "\alpha'"] \\
\mu_N\times\mathbf{Z}/N \arrow[r, "\sim"] & \mu_N\times\mathbf{Z}/N
\end{tikzcd}$$

where the lower horizontal arrow is the automorphism

$$(\zeta, a)\mapsto(\zeta^p, a)$$

of $\mu_N \times \mathbf{Z}/N$. The morphism ϕ is then the classifying map for the curve with level N structure $(E'/U, \alpha')$. It maps $\underline{\infty}$ to itself, and locally at $\underline{\infty}$ is given by

$$\phi^*(q^{1/N}) = q^{p/N}.$$

The ϕ^*-linear endomorphism F_ϕ of $\hat{\psi}_0^* \mathscr{E} = \mathscr{E} \otimes \widehat{\mathbf{Z}_p((q^{1/N}))}$ is described explicitly in [K1] A.2.2; it has the form (3.2.1). Since $\{\omega, \xi\}$ extends to a basis of $\hat{\psi}^* \mathscr{E}$, the action of F_ϕ extends across $\underline{\infty}$, and by virtue of the action of G_N, across all of $\widehat{Z(N)}$; and $F_\phi \otimes \mathbf{Q}$ is an isomorphism, since the endomorphism "multiplication by p" on E_{univ} (which induces multiplication by p on $\mathscr{E}$) factors through ϕ.

3.3. Now assume that we have $(X, h, \underline{\infty}')$ as in §2.9. For $p \nmid 2M$, it follows from the above and §7.1 of [S] that $h^* \hat{\mathscr{E}}$ and its symmetric powers define F-crystals on $\hat{X}$ with logarithmic singularities along $\hat{Z}$. To calculate their form near ∞', define γ_p by the conditions

$$\gamma_p \in 1 + p\mathbf{Z}_p, \quad \gamma_p^v = D^{p-1} \tag{3.3.1}$$

(cf. (2.9.1)). The existence and uniqueness of γ_p are assured by Hensel's lemma. Then

$$\phi^*\colon\ t \longmapsto \gamma_p t^p$$

defines an extension of the endomorphism ϕ of 3.2 to $\mathbf{Z}_p[\![t]\!]$.

There are then canonical endomorphisms F of $\mathbf{H}^1(\hat{X}, \widehat{\Omega^{\cdot}(\mathscr{E}_k)})$ and $H^1(\hat{\psi}'^* \Omega^{\cdot}(\mathscr{E}_k))$. By the fundamental theorem of a proper morphism, the first of these is $L_k(X, \mathbf{Z}_p)$; the second is $L_k^\infty(X, \mathbf{Z}_p)$, and the action of F on it can be calculated using the lifting ϕ. If $p > k$, then from the description (3.2.1) of F_ϕ, and the construction of the isomorphism

$$L_k^\infty(X, \mathbf{Z}_p) \xrightarrow{\sim} \operatorname{coker}(\partial^{k+1}\colon\ t \cdot \mathbf{Z}_p[\![t]\!] \to t \cdot \mathbf{Z}_p[\![t]\!]),$$

we find

$$F\left(\sum a_n t^n\right) \equiv p^{k+1} \sum a_n \gamma_p^n t^{np} \quad (\operatorname{mod} \operatorname{Im} \partial^{k+1}). \tag{3.3.2}$$

3.4. **Proposition.** *Suppose that $p > k+1$. Then*

$$F(S_{k+2}(X, \mathbf{Z}_p)) \subseteq p^{k+1} L_k(X, \mathbf{Z}_p).$$

Proof. For any lifting ϕ of Frobenius satisfying (3.2.2)

$$F_\phi(\phi^* \omega) \subseteq p \mathscr{E}$$

and so, for the filtration (2.7.1) of $\mathscr{E}_k$, we have

$$F_\phi(\phi^* F^i_{\mathrm{Hdg}} \mathscr{E}_k) \subseteq p^i \mathscr{E}_k.$$

Let now $x \in H^0(X, \omega^k \otimes \Omega_X^1 \otimes \mathbf{Z}_p) = S_{k+2}(X, \mathbf{Z}_p)$. Then if ϕ, ϕ' are two liftings of Frobenius to an open $U \subseteq \hat{X}$, we have, in the notations of [S],

$$F_\phi(\phi^* x) \in p^k \mathscr{E}_k \otimes \Omega_X^1 \quad \text{and} \quad L(\phi, \phi')(\phi^* x) \in p^{k+1} \mathscr{E}_k$$

(the second inclusion following from the formula (2.1.1) of [S] together with the transversality (2.7.2) and the condition $p > k+1$). Thus the Čech cocycle representing $F(x)$ is divisible by p^{k+1} in the group of 1-cochains, and so $F(x) \in p^{k+1} L_k(X, \mathbf{Z}_p)$ since the cochain groups in question are torsion-free.

3.5. It follows from the above and Theorem 2.12 that F takes ${}^{(p)}L_k(X, \mathbf{Z}_p)$ into itself. This is also a special case of § 7.3 of [S], where it is also shown that if x is an element of ${}^{(p)}L_k^{\infty}(X, \mathbf{Z}_p)$ which is in the image of ${}^{(p)}L_k(X, \mathbf{Z}_p)$, then $F(x)$ can be calculated from the formula (3.3.2) applied to the description 2.12.ii) of ${}^{(p)}L_k^{\infty}(X, \mathbf{Z}_p)$.

The action of G on $L_k(X, \mathbf{Z}_p)$ commutes with F. Indeed, by § 7.2 of [S], it suffices to show that, if A is a finite étale $\mathbf{Z}_p$-algebra, and if ϕ, ϕ' are two liftings of Frobenius to an open subscheme of $\hat{X} \otimes A$, satisfying $g \circ \phi = \phi' \circ g$ for some $g \in G(A)$, then $g \circ F_{\phi} = F_{\phi'} \circ g$ on $\mathscr{E}$. This holds over Y, by the functionality of Frobenius acting on crystalline cohomology, and therefore over all of X as Y is dense in it. We may therefore summarise our findings as follows.

3.6. **Theorem.** *Let p be prime, with $p \nmid M$, $p > k+1$. Then there is a canonical endomorphism F of $L_k(X, \mathbf{Z}_p)$, which takes ${}^{(p)}L_k(X, \mathbf{Z}_p)$ into $p^{k+1} L_k(X, \mathbf{Z}_p) \subseteq {}^{(p)}L_k(X, \mathbf{Z}_p)$, and which commutes with the action of G. If ρ denotes the canonical map*

$$\rho\colon {}^{(p)}L_k(X, \mathbf{Z}_p) \to {}^{(p)}L_k^{\infty}(X, \mathbf{Z}_p) \simeq \frac{t \cdot \mathbf{Z}_p[[t]]}{(p\,\partial)^{k+1}(t \cdot \mathbf{Z}_p[[t]])}$$

and if $x \in {}^{(p)}L_k(X, \mathbf{Z}_p)$, with

$$\rho(x) \equiv \sum a_n t^n \qquad (\operatorname{mod} \operatorname{Im}(p\,\partial)^{k+1})$$

then

$$\rho(F x) \equiv \sum p^{k+1} a_n \gamma_p^n t^{np} \qquad (\operatorname{mod} \operatorname{Im}(p\,\partial)^{k+1}).$$

3.7. *Remark.* The statement that F commutes with the action of G means that if $g \in G(\mathbf{Z}_p^{\mathrm{nr}})$, then g commutes with the *semilinear* extension of F to $L_k(X, \mathbf{Z}_p^{\mathrm{nr}}) = L_k(X, \mathbf{Z}_p) \otimes \mathbf{Z}_p^{\mathrm{nr}}$, cf. 4.4 below.

4. *l*-adic theory

In this section l denotes a prime number; all cohomology is étale (l-adic) unless otherwise indicated. We summarise some of the constructions of [D1], to which, together with (3.7.1) of [D4], the reader is referred for further details. As before, we assume $N \geqq 3$.

4.1. Write E^0 for the restriction of E_{univ} to $Y(N)[1/Ml]$; we have a diagram

$$\begin{tikzcd}
E^0 \arrow[d, "f^0"] & Y[1/l] \arrow[r, "j'"] \arrow[dl, "h'"'] & X[1/l] \arrow[dl, "h"'] \arrow[dd, "c"] \\
Y(N)[1/Ml] \arrow[r, hook, "j"] & X(N)[1/Ml] & \\
 & & \operatorname{Spec} \mathbf{Z}[1/Ml].
\end{tikzcd}$$

The $\mathbf{Q}_l$-sheaf $\mathscr{F}_k := \mathrm{Sym}^k(R^1 f_*^0 \mathbf{Q}_l)$ is a smooth sheaf of rank $k+1$ on $Y(N)[1/Ml]$, pure of weight k, and tamely ramified along $Z(N)[1/Ml]$. The Poincaré duality

$$\Lambda^2 \mathscr{F}_k \xrightarrow{\sim} \mathbf{Q}_l(-1)$$

gives a perfect pairing

$$\mathscr{F}_k \times \mathscr{F}_k \to \mathbf{Q}_l(-k). \tag{4.1.1}$$

The local monodromy of $\mathscr{F}_k$ at $Z(N)$ is unipotent, and so

$$h^* j_* \mathscr{F}_k = j'_* h'^* \mathscr{F}_k.$$

By the theory of the Tate curve, the restriction to $Z(N)[1/Ml]$ of $j_* \mathscr{F}_k$ is the constant sheaf $\mathbf{Q}_l$.

4.2. The sheaf $R^i c_*(h^* j_* \mathscr{F}_k)$ is zero if $i \neq 1$; and

$$\mathscr{W}_k := R^1 c_*(h^* j_* \mathscr{F}_k)$$

is a smooth $\mathbf{Q}_l$-sheaf on $\mathbf{Z}[1/Ml]$, whose formation is compatible with arbitrary base-change, and which is pure of weight $k+1$. By the local duality $\mathbf{D} j_* \mathscr{F}_k = j_* \mathbf{D} \mathscr{F}_k$ and (4.1.1) there is a perfect pairing

$$\mathscr{W}_k \times \mathscr{W}_k \to \mathbf{Q}_l(-k-1) \tag{4.2.1}$$

which is alternating for k even, symmetric for k odd. Let $\mathscr{W}_k(\bar{\mathbf{Q}})$ denote the $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-module associated to $\mathscr{W}_k$.

4.3. The subgroup scheme $G \subseteq G_N$ acts on $\mathscr{W}_k$, whence there is a $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$-equivariant action

$$G(\bar{\mathbf{Q}}) \times \mathscr{W}_k(\bar{\mathbf{Q}}) \to \mathscr{W}_k(\bar{\mathbf{Q}}).$$

The pairing (4.2.1) is invariant with respect to this action.

Let p be a prime which does not divide $2M$. Then the representation $\mathscr{W}_k(\bar{\mathbf{Q}})$ of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ is unramified at p. Choose a prime $\mathfrak{p}$ of $\bar{\mathbf{Q}}$ over p, and let $\mathrm{Frob}_{\mathfrak{p}} = \sigma_{\mathfrak{p}}^{-1} \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be a geometric Frobenius element at $\mathfrak{p}$. The choice of $\mathfrak{p}$ defines isomorphisms

$$G(\bar{\mathbf{Q}}) \xrightarrow{\sim} G(\bar{\mathbf{F}}_p), \qquad \mathscr{W}_k(\bar{\mathbf{Q}}) \xrightarrow{\sim} \mathscr{W}_k(\bar{\mathbf{F}}_p)$$

taking $\mathrm{Frob}_{\mathfrak{p}}$ to the geometric Frobenius F_p.

4.4. **Proposition.**

$$\det(1 - TF_p \colon \mathscr{W}_k(\bar{\mathbf{F}}_p)^G) = \det(1 - TF \colon L_k(X, \mathbf{Z}_p)^G) \in \mathbf{Z}[T].$$

Proof. The coefficients of the polynomial on the left-hand side are algebraic integers. The l-adic Frobenius F_p is an automorphism of $\mathscr{W}_k(\bar{\mathbf{F}}_p)$, and the p-adic Frobenius F is injective on $L_k(X, \mathbf{Z}_p)$ (cf. Theorem 5.2 of [S]). Therefore 4.4 is equivalent to the statement: for every $s \geqq 1$

$$\mathrm{Tr}(F_p^{-s} \colon \mathscr{W}_k(\bar{\mathbf{F}}_p)^G) = \mathrm{Tr}(F^{-s} \colon L_k(X, \mathbf{Q}_p)^G) \in \mathbf{Q}.$$

By the pairing (4.2.1) we may rewrite this as

$$\mathrm{Tr}(F_p^s\colon\ \mathscr{W}_k(\bar{\mathbf{F}}_p)^G)=p^{(k+1)s}\,\mathrm{Tr}(F^{-s}\colon\ L_k(X,\mathbf{Q}_p)^G)\in\mathbf{Q}.$$

Using the projector $\frac{1}{\# G}\sum g$, we see that 4.4 is a consequence of:

For all $s\geqq 1$ and every $g\in G(\bar{\mathbf{F}}_p)=G(\mathbf{Z}_p^{\mathrm{nr}})$,

$$\mathrm{Tr}(F_p^s\circ g^{-1}\colon\ \mathscr{W}_k(\bar{\mathbf{F}}_p))=p^{(k+1)s}\,\mathrm{Tr}((F\otimes\mathrm{id})^{-s}\circ g\colon\ L_k(X,\mathbf{Q}_p^{\mathrm{nr}}))\in\mathbf{Z}. \tag{4.4.1}$$

Here $F\otimes\mathrm{id}$ denotes the linear extension of F to

$$L_k(X,\mathbf{Q}_p^{\mathrm{nr}})=L_k(X,\mathbf{Z}_p)\underset{\mathbf{Z}_p}{\otimes}\mathbf{Q}_p^{\mathrm{nr}}.$$

First consider the case $g=1$. The left hand member of (4.4.1) can be calculated by the Lefchetz fixed point theorem in l-adic cohomology ([SGA4$\frac{1}{2}$] Rapport 3.2), since

$$\mathscr{W}_k(\bar{\mathbf{F}}_p)=H^1(X\otimes\bar{\mathbf{F}}_p,\ h^*j_*\mathscr{F}_k)$$

and the H^0 and H^2 of $h^*j_*\mathscr{F}_k$ are zero. The local terms are of two sorts:

for each $x\in Y(\mathbf{F}_q)$, where $q=p^s$, there is a term

$$\mathrm{Tr}(F_x\colon\ (h^*j_*\mathscr{F}_k)_x)=\mathrm{Tr}(F_q\colon\ \mathrm{Sym}^k H^1(E_{h(x)}\otimes\bar{\mathbf{F}}_q,\mathbf{Q}_l))\in\mathbf{Z} \tag{4.4.2}$$

where $E_{h(x)}$ is the fibre of $E_{\mathrm{univ}}\to X(N)$ at $h(x)$;

for each $x\in Z(\mathbf{F}_q)$, the contribution from the local invariants is $\mathrm{Tr}(F_q\colon\ \mathbf{Q}_l)=1$.

By Theorem 5.2 of [S], and the vanishing of $\mathbf{H}^i(X,\Omega^{\boldsymbol{\cdot}}(\mathscr{E}_k))$ for $i\neq 1$ (cf. the proof of 2.7 above), the right hand member of (4.4.1) is the sum of terms:

for $x\in Y(\mathbf{F}_q)$,

$$p^{ks}\,\mathrm{Tr}(F_x^{-s}\colon \mathrm{Sym}^k\mathscr{E}_k\otimes\mathbf{Q})=q^k\,\mathrm{Tr}(F^{-s}\colon \mathrm{Sym}^k H^1_{\mathrm{cris}}(E_{h(x)}/W(\mathbf{F}_q))\otimes\mathbf{Q})$$

which is the same as (4.4.2);

for $x\in Z(\mathbf{F}_q)$,

$$\mathrm{Tr}(p^{ks}F_x^{-s}\colon\ (\mathrm{Sym}^k\mathscr{E}_x/\mathrm{Im}\,\mathscr{R}_x)\otimes\mathbf{Q})$$

where $\mathscr{R}_x\colon \mathrm{Sym}^k\mathscr{E}_x\to\mathrm{Sym}^k\mathscr{E}_x$ is the residue map.

Since

$$\mathrm{Sym}^k\mathscr{E}_x=\bigoplus_{r=0}^{k} W(\mathbf{F}_q)\cdot\omega^{k-r}\cdot\xi^r$$

and

$$\mathscr{R}_x(\omega^{k-r}\cdot\xi^r)=(k-r)\cdot\omega^{k-r-1}\cdot\xi^{r+1}$$

the cokernel of $\mathscr{R}_x$ is one-dimensional, spanned by ω^k, and so this local contribution is 1, as $F(\omega)=p\cdot\omega$. Thus (4.4.1) holds.

When $1\neq g\in G(\bar{\mathbf{F}}_p)$, we resort to the usual twisting argument, as follows:

There are twisted forms $X^{(g)}$, $X(N)^{(g)}$ of X, $X(N)$ over $R=W(\mathbf{F}_q)$, where $q=p^s$, such that

$$X^{(g)}(\mathbf{F}_q)=\{x\in X(\bar{\mathbf{F}}_q)\colon\ x^q=g(x)\}.$$

$X(N)^{(g)}$ has a similar property, and may be defined in a modular fashion as follows. Let $V^{(g)}/R$ denote the twisted form of $\mu_N \times \mathbf{Z}/N$ over R, defined by the 1-cocycle $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/R\otimes\mathbf{Q})\to G_N$ whose value on the Frobenius is g. Then $X(N)^{(g)}$ represents the functor on R-schemes S

"isomorphism classes of generalised elliptic curves E/S, whose geometric fibres are smooth or N-gons, together with an isomorphism $\alpha\colon {}_NE \xrightarrow{\sim} V^{(g)}$ of determinant one".

There is a universal curve $E^{(g)}_{\mathrm{univ}} \to X(N)^{(g)}$, and objects $\mathscr{E}^{(g)}$, $\mathscr{F}_k^{(g)}$, $\mathscr{W}_k^{(g)}$, $L_k^{(g)}, \ldots$ constructed in the same way as $\mathscr{E}$, $\mathscr{F}_k$, etc. We then have

$$\mathrm{Tr}(F_p^s \circ g^{-1}\colon \mathscr{W}_k(\bar{\mathbf{F}}_p)) = \mathrm{Tr}(F_q\colon \mathscr{W}_k^{(g)}(\bar{\mathbf{F}}_q)).$$

The method described in the case $g=1$ then shows that

$$\mathrm{Tr}(F_q\colon \mathscr{W}_k^{(g)}(\bar{\mathbf{F}}_q)) = q^{k+1}\,\mathrm{Tr}(F'^{-s}\colon L_k^{(g)}(X,R)\otimes\mathbf{Q}) \tag{4.4.3}$$

where F' is the canonical Frobenius endomorphism of $L_k^{(g)}(X,R)$. If σ denotes the canonical lifting of the p-power Frobenius of $\bar{\mathbf{F}}_p$ to $\mathbf{Z}_p^{\mathrm{nr}}$, then F' is σ-linear (and so F'^s is linear).

Now $\mathcal{O}_{X^{(g)}}$ (or rather its pullback to $X\otimes\mathbf{Z}_p^{\mathrm{nr}}$) can be identified with

$$\{b\in\mathcal{O}_X\otimes\mathbf{Z}_p^{\mathrm{nr}}\colon g^*(\mathrm{id}\otimes\sigma^s)(b)=b\}$$

and similarly for $E^{(g)}$. Thus

$$L_k^{(g)}(X,R\otimes\mathbf{Q}) = \{x\in L_k(X,\mathbf{Q}_p^{\mathrm{nr}})\colon g\circ(\mathrm{id}\otimes\sigma^s)(x)=x\}. \tag{4.4.4}$$

If $F\otimes\sigma$ is the semilinear extension of F to

$$L_k(X,\mathbf{Q}_p^{\mathrm{nr}}) = L_k(X,\mathbf{Z}_p)\otimes\mathbf{Q}_p^{\mathrm{nr}}$$

then (4.4.4) identifies F' with the restriction of $F\otimes\sigma$ to $L_k^{(g)}(X,R\otimes\mathbf{Q})$. Thus if $x\in L_k^{(g)}(X,R\otimes\mathbf{Q})$

$$\begin{aligned} F'^{-s}(x) &= (F\otimes\sigma)^{-s}(x)\\ &= (F\otimes\sigma)^{-s}\circ g\circ(\mathrm{id}\otimes\sigma^s)(x)\\ &= g\circ(F\otimes\sigma)^{-s}\circ(\mathrm{id}\otimes\sigma^s)(x)\\ &= g\circ(F\otimes\mathrm{id})^{-s}(x)\end{aligned}$$

and so the right hand members of (4.4.1) and (4.4.3) are equal, as required.

5. Atkin-Swinnerton-Dyer Congruences

5.1. Let Γ be a subgroup of $PSL_2(\mathbf{Z})$ of finite index, and let μ denote the width of the cusp $i\infty$, so that

$$\Gamma\cap\left\{\pm\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\} = \left\langle\pm\begin{pmatrix}1&\mu\\0&1\end{pmatrix}\right\rangle.$$

For $k\geq 0$, write $S_{k+2}(\Gamma)$ for the space of holomorphic cusp forms on Γ of (even) weight $k+2$.

Consider the compactified quotient space $\Gamma\backslash\mathfrak{H}^*$, and the canonical map

$$\Gamma\backslash\mathfrak{H}^* \to \Gamma(1)\backslash\mathfrak{H}^*.$$

We will say that Γ is defined over $\mathbf{Q}$ if there exist

i) a nonsingular projective curve $V/\mathbf{Q}$;
ii) a finite morphism $\pi\colon V \to \mathbf{P}^1_{\mathbf{Q}}$;
iii) a point $e \in V(\mathbf{Q})$; and
iv) an isomorphism

$$\Xi\colon \Gamma\backslash\mathfrak{H}^* \xrightarrow{\sim} V(\mathbf{C})$$

such that $\Xi(i\infty) = e$, and the diagram

$$\begin{tikzcd}
\Gamma\backslash\mathfrak{H}^* \arrow[r] \arrow[d, "\Xi", "\simeq"'] & \Gamma(1)\backslash\mathfrak{H}^* \arrow[d, "j", "\simeq"'] \\
V(\mathbf{C}) \arrow[r, "\pi_{\mathbf{C}}"] & \mathbf{P}^1(\mathbf{C})
\end{tikzcd}$$

commutes (where here j is the usual modular invariant of level 1).

Remark. We do not assert the uniqueness of such a system (V, π, e, Ξ).

From now on we assume that Γ is defined over $\mathbf{Q}$, and fix a quadruple (V, π, e, Ξ) as above. As in the rest of the paper, we restrict to the case $k > 0$.

5.2. **Proposition.** *For some* $M \geqq 1$ *and* $\gamma_\infty \in \mathbf{C}$, *with* $\gamma_\infty^\mu = C \in \mathbf{Z}[1/M]^*$, *there exists, for each* k, *a basis of the space* $S_{k+2}(\Gamma)$ *comprised of forms* f *with Fourier expansion*

$$\tilde{f} = \sum_{n \geqq 1} a(n) \cdot \exp(2\pi i n \tau/\mu) \tag{5.2.1}$$

such that $b(n) := \gamma_\infty^n \cdot a(n) \in \mathbf{Z}[1/M]$.

Proof. a) We first assume: $\Gamma \subseteq \pm\Gamma(N)$ for some $N \geqq 3$, and π factors

$$\begin{tikzcd}
V \arrow[rr, "\pi"] \arrow[dr, "h_{\mathbf{Q}}"'] & & \mathbf{P}^1_{\mathbf{Q}} \\
& X(N)_{\mathbf{Q}} \arrow[ur] &
\end{tikzcd} \tag{5.2.2}$$

such that $h_{\mathbf{C}} \circ \Xi$ is the quotient map

$$\Gamma\backslash\mathfrak{H}^* \to \Gamma(N)\backslash\mathfrak{H}^*. \tag{5.2.3}$$

Let M be divisible by N, and write X for the normalisation of $X(N)[1/M]$ in V. Then $X_{\mathbf{Q}} = V$, and $h_{\mathbf{Q}}$ extends to a finite morphism

$$h\colon X \to X(N)[1/M];$$

e extends to a section $\underline{\infty}'$ of X over $\mathbf{Z}[1/M]$, such that $\underline{\infty} = h \circ \underline{\infty}'$.

Since $N \geqq 3$, the covering (5.2.3) is unramified away from the cusps, whence $h_{\mathbf{Q}}$ is étale over $Y(N)_{\mathbf{Q}}$. Then for suitably chosen $M \geqq 1$, h will be étale over $Y(N)[1/M]$. We fix such an M.

We are now in the situation of 2.9, and may therefore choose a uniformiser t along $\underline{\infty}'$ with

$$D \cdot t^{\nu} = q^{1/N}, \qquad D \in \mathbf{Z}[1/M]^*$$

where $\nu = \mu/N$. The uniformisation Ξ of $X(\mathbf{C})$ identifies t with

$$\gamma_{\infty}^{-1} \exp(2\pi i \tau/\mu)$$

for some $\gamma_{\infty} \in \mathbf{C}$ with $\gamma_{\infty}^{\nu} = D$.

Standard GAGA arguments (cf. [D–R] VII.4.6, [K1] A1) now identify cusp forms on Γ with sections of $\omega^k \otimes \Omega_X^1 \otimes \mathbf{C}$, giving a commutative diagram

$$\begin{tikzcd}
S_{k+2}(\Gamma) \arrow[r, "\sim", "\text{GAGA}"'] \arrow[d, "\text{Fourier expansion}"] & S_{k+2}(X, \mathbf{C}) \arrow[d, "t\text{-expansion}"] \\
\mathbf{C}[\![q^{1/\mu}]\!] \arrow[r, "\sim", "t = \gamma_{\infty} q^{1/\mu}"'] & \mathbf{C}[\![t]\!].
\end{tikzcd}$$

From the "t-expansion principle" and the base-change isomorphism

$$S_{k+2}(X, \mathbf{C}) = S_{k+2}(X, \mathbf{Z}[1/M]) \otimes \mathbf{C}$$

of 2.9, the proposition follows, taking $C = D^N$.

b) We now suppose only that Γ is defined over $\mathbf{Q}$. Choose $N \geqq 3$ such that

$$(N, \mu) = 1 \quad \text{and} \quad \pm\Gamma(N) \cdot \Gamma = PSL_2(\mathbf{Z}).$$

Let $\Delta = \Gamma \cap \pm\Gamma(N)$. Then Δ satisfies the conditions (5.2.2) above. Indeed, let W be the normalisation of the fibre product

$$V \underset{\mathbf{P}^1_{\mathbf{Q}}}{\times} X(N)_{\mathbf{Q}}. \tag{5.2.4}$$

By the choice of N, there is a canonical isomorphism

$$\Delta \backslash \mathfrak{H}^* \xrightarrow{\sim} W(\mathbf{C}) \tag{5.2.5}$$

which is compatible with Ξ, with respect to the obvious maps.

The ramification degrees of the coverings V and $X(N)_{\mathbf{Q}}$ of $\mathbf{P}^1_{\mathbf{Q}}$ at e and $\underline{\infty}$ respectively are coprime (they are μ and N), and there is therefore a unique point $\tilde{e} \in W(\mathbf{Q})$ having e and $\underline{\infty}$ as images.

As in a), we may choose M and a triple $(X, h, \underline{\infty}')$ with $X_{\mathbf{Q}} = W$. Since the width of the cusp $i\infty$ of Δ is $N\mu$, the uniformiser t along $\underline{\infty}'$ is identified under (5.2.5) with

$$\delta_{\infty}^{-1} \exp(2\pi i \tau/N\mu)$$

where $\delta_{\infty} \in \mathbf{C}$, $\delta_{\infty}^{\mu} = D \in \mathbf{Z}[1/M]^*$.

The group scheme G_N acts on the fibre product (5.2.4) via the second factor, and therefore acts on X, covering its action on $X(N)$. We have

$$S_{k+2}(\Gamma) = S_{k+2}(\Delta)^{\Gamma/\Delta} = S_{k+2}(X, \mathbf{C})^{G_N} = S_{k+2}(X, \mathbf{Z}[1/M])^{G_N} \otimes \mathbf{C}.$$

The t-expansion of an element of $S_{k+2}(\Gamma)$ is contained in $\mathbf{C}[\![t^N]\!]$, and since t^N identifies with $\delta_\infty^{-N}\exp(2\pi i\tau/\mu)$, the proposition follows, with $\gamma_\infty=\delta_\infty^N$.

5.3. We associate to Γ and k a system of l-adic representations

$$\rho_l\colon \operatorname{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})\to GL_{2d}(\mathbf{Q}_l),$$

where $d=\dim S_{k+2}(\Gamma)$, as follows. If Γ satisfies the conditions of a) above, ρ_l is the representation $\mathscr{W}_k(\bar{\mathbf{Q}})$ of 4.2. In the general case, we choose N as in b), and ρ_l is taken to be $\mathscr{W}_k(\bar{\mathbf{Q}})^{G_N}$.

In any case, ρ_l is unramified away from Ml, and if Frob_p is a (geometric) Frobenius element for $p\nmid Ml$, the characteristic polynomial of $\rho_l(\mathrm{Frob}_p)$ can be written, by the results of § 4, as

$$H_p(T)=\sum_{r=0}^{2d} A_r(p)\cdot T^{2d-r}=\prod_{i=1}^{d}(T-\alpha_i)(T-p^{k+1}/\alpha_i)\in\mathbf{Z}[T]$$

with $|\alpha_i|=p^{(k+1)/2}$. We therefore have

$$\begin{aligned} &A_{2d-r}=p^{(k+1)(d-r)}\cdot A_r \qquad (0\leqq r\leqq 2d)\\ &A_0=1, \qquad A_{2d}=p^{(k+1)d}. \end{aligned} \tag{5.3.1}$$

With C as in the proposition, define, for primes $p\nmid M$, constants $\gamma_p\in 1+p\mathbf{Z}_p$ with $\gamma_p^\mu=C^{p-1}$, as in (3.3.1).

5.4. **Theorem.** *Let $f\in S_{k+2}(\Gamma)$ have Fourier expansion as in* (5.2.1) *above. Then if $p\nmid M$, $p>k+1$,*

$$\begin{aligned} &\operatorname{ord}_p(a(n\,p^d)+A_1(p)\,a(n\,p^{d-1})+\ldots+A_{2d-1}(p)\,a(n/p^{d-1})+A_{2d}(p)\,a(n/p^d))\\ &\quad\geqq(k+1)(\operatorname{ord}_p n+1) \end{aligned}$$

for all $n\geqq 1$.

Here the left hand expression is interpreted as follows. Formally writing

$$a(n)=b(n)\cdot C^{-n/\mu}$$

(and writing A_r for $A_r(p)$ throughout) we have

$$\begin{aligned} \sum_{r=0}^{2d} A_r\,a(n\,p^{d-r})&=\sum_{r=0}^{2d} A_r\cdot C^{-np^{d-r}/\mu}\cdot b(n\,p^{d-r})\\ &=C^{-np^d/\mu}\sum_{r=0}^{2d} A_r\cdot C^{n(p^d-p^{d-r})/\mu}\cdot b(n\,p^{d-r}). \end{aligned}$$

Since $\operatorname{ord}_p(C)=0$, we define

$$\operatorname{ord}_p\left(\sum_{r=0}^{2d} A_r\,a(n\,p^{d-r})\right):=\operatorname{ord}_p\left(\sum_{r=0}^{2d} A_r\cdot\gamma_p^{n(p^d-p^{d-r})/(p-1)}\cdot b(n\,p^{d-r})\right)$$

with the usual convention that $a(n)=b(n)=0$ if $n\notin\mathbf{N}$.

5.5. *Remark.* If $d=1$, the left hand expression is

$$\operatorname{ord}_p(a(np)-A(p)\cdot a(n)+p^{k+1}a(n/p))$$

where $H_p(T)=T^2-A(p)\cdot T+p^{k+1}$. This is the first type of congruence originally discovered by Atkin and Swinnerton-Dyer.

5.6. **Theorem.** *Suppose that* $p\nmid M$, $p>k+1$, *and that* $H_p(T)$ *is* ordinary, *so that*

$$H_p(T)=\prod_{i=1}^{d}(T-u_i)(T-p^{k+1}/u_i)$$

with $u_i\in\bar{\mathbf{Q}}_p$, $\operatorname{ord}_p(u_i)=0$.

Let $\{f_1,\dots,f_d\}$ *be a basis for* $S_{k+2}(\Gamma)$ *as in Proposition* 5.2, *with*

$$f_i=\sum a_i(n)\cdot q^{n/\mu}=\sum b_i(n)\cdot t^n,\qquad b_i(n)\in\mathbf{Z}[1/M].$$

Then for some matrix $(B_{ij}(p))\in M_d(\mathbf{Z}_p)$, *and each* $n\geqq 1$,

$$\begin{aligned}\operatorname{ord}_p(a_i(np)-\sum_j B_{ij}(p)\cdot a_j(n)):&=\operatorname{ord}_p(b_i(np)-\gamma_p^n\sum_j B_{ij}(p)\cdot b_j(n))\\&\geqq(k+1)(\operatorname{ord}_p n+1);\end{aligned}$$

and the eigenvalues of $(B_{ij}(p))$ *are the non-unit roots* $\{p^{k+1}/u_i\}$.

5.7. *Proof of Theorems 5.4 and 5.6.* a) We assume that the additional hypotheses (5.2.3) hold.

View the forms f, f_i as elements of $S_{k+2}(X,\mathbf{Z}[1/M])\hookrightarrow S_{k+2}(X,\mathbf{Z}_p)$ as in the proof of 5.2. By 3.6 and 4.4, taking $G=1$, the image of f in ${}^{(p)}L_k^\infty(X,\mathbf{Z}_p)$ is annihilated by $H_p(F)$; this translates into the t-expansion condition

$$\begin{aligned}\sum_{n\geqq 1}c(n)\cdot t^n:&=\sum_{r=0}^{2d}A_r\Big(\sum_{n\geqq 1}b(n)\cdot p^{(k+1)(2d-r)}\cdot\gamma_p^{n(1+p+\dots+p^{2d-r-1})}\cdot t^{np^{2d-r}}\Big)\\&\in\operatorname{Im}(p\,\partial)^{k+1}.\end{aligned}$$

Thus

$$\begin{aligned}c(n)&=\sum_{r=0}^{2d}A_r\cdot p^{(k+1)(2d-r)}\cdot\gamma_p^{n(1-p^{-2d+r})/(p-1)}\cdot b(np^{-2d+r})\\&=p^{(k+1)d}\sum_{r=0}^{2d}A_r\cdot\gamma_p^{n(1-p^{-r})/(p-1)}\cdot b(n/p^r)\end{aligned}\tag{5.7.1}$$

by (5.3.1). Since

$$\sum c(n)\cdot t^n\in\operatorname{Im}\left(p\,t\frac{d}{dt}\right)^{k+1}$$

we have $c(n)\in(np)^{k+1}\mathbf{Z}_p$ whence

$$\sum_{r=0}^{2d}A_r\cdot\gamma_p^{n(1-p^{-r})/(p-1)}\cdot b(n/p^r)\in(np^{1-d})^{k+1}\mathbf{Z}_p.$$

This gives no information if $p^d \not| n$, so replacing n by $p^d n$,

$$\sum_{r=0}^{2d} A_r \cdot \gamma_p^{n(p^d - p^{d-r})/(p-1)} \cdot b(n p^{d-r}) \in (n p)^{k+1} \mathbf{Z}_p$$

which is Theorem 5.4.

Now suppose that $H_p(T)$ is ordinary. Since

$$F(S_{k+2}(X, \mathbf{Z}_p)) \subseteq p^{k+1} L_k(X, \mathbf{Z}_p)$$

we have

$$L_k(X, \mathbf{Z}_p) = S_{k+2}(X, \mathbf{Z}_p) \oplus U \tag{5.7.2}$$

where U is F-stable, and F is invertible on U, with eigenvalues $u_1, \dots, u_d$. Then F is divisible by p^{k+1} on the quotient $L_k(X, \mathbf{Z}_p)/U$, and we may write

$$F(f_i) \equiv p^{k+1} \sum C_{ij} \cdot f_j \quad (\operatorname{mod} p^{k+1} U)$$

where (C_{ij}) is invertible, with eigenvalues $u_1^{-1}, \dots, u_d^{-1}$. Then

$${}^{(p)}L_k(X, \mathbf{Z}_p) = S_{k+2}(X, \mathbf{Z}_p) \oplus p^{k+1} \cdot U.$$

Writing ρ for the map ${}^{(p)}L_k \to {}^{(p)}L_k^{\infty}$, we have

$$\rho(p^{k+1} \cdot U) = 0.$$

Indeed, it suffices to show that if

$$u \in \bar{\mathbf{Z}}_p^*, \quad \text{and} \quad g \in t \cdot \bar{\mathbf{Z}}_p[\![t]\!]$$

then

$$(F - u) g \in \operatorname{Im}(p\,\partial)^{k+1} \Leftrightarrow g \in \operatorname{Im}(p\,\partial)^{k+1}$$

(F being defined on $t \cdot \bar{\mathbf{Z}}_p[\![t]\!]$ by the formula (3.3.2)); and if

$$g = \sum_{n \geqq 1} d(n) \cdot t^n$$

then

$$(F - u) g \in \operatorname{Im}(p\,\partial)^{k+1}$$

implies

$$p^{k+1} \cdot \gamma_p^{n/p} \cdot d(n/p) - u \cdot d(n) \in (n p)^{k+1} \bar{\mathbf{Z}}_p$$

whence $d(n) \in (n p)^{k+1} \bar{\mathbf{Z}}_p$ as required.

Therefore $F(\tilde{f}_i) - p^{k+1} \sum C_{ij} \cdot \tilde{f}_j \in \operatorname{Im}(p\,\partial)^{k+1}$ whence, writing $(B_{ij}) = (C_{ij})^{-1}$,

$$\begin{aligned} & p^{k+1} \cdot \tilde{f}_i - \sum B_{ij} \cdot F(\tilde{f}_j) \in \operatorname{Im}(p\,\partial)^{k+1} \\ & \quad \Rightarrow \; p^{k+1} \cdot b_i(n) - \sum B_{ij} \cdot p^{k+1} \cdot \gamma_p^{n/p} \cdot b_j(n/p) \in (p\,n)^{k+1} \mathbf{Z}_p \\ & \quad \Rightarrow \; b_i(n\,p) - \sum B_{ij} \cdot \gamma_p^{n} \cdot b_j(n) \in (p\,n)^{k+1} \mathbf{Z}_p \end{aligned}$$

proving Theorem 5.6 in this case.

5.8. *Remark.* Under the hypotheses of Theorem 5.6 we have (in an obvious notation)

$$\mathbf{b}(n\,p)-\gamma_p^n\cdot\mathbf{B}\cdot\mathbf{b}(n)\equiv\mathbf{0}\qquad(\operatorname{mod}(p\,n)^{k+1}\,\mathbf{Z}_p)$$

and

$$\mathbf{C}\cdot\mathbf{b}(n)-\gamma_p^{n/p}\cdot\mathbf{b}(n/p)\equiv\mathbf{0}\qquad(\operatorname{mod} n^{k+1}\,\mathbf{Z}_p)$$

whence

$$\mathbf{b}(n\,p)-\gamma_p^n\cdot\mathbf{A}\cdot\mathbf{b}(n)+p^{k+1}\cdot\gamma_p^{n(1+1/p)}\cdot\mathbf{b}(n/p)\equiv\mathbf{0}\qquad(\operatorname{mod}(p\,n)^{k+1}\,\mathbf{Z}_p)$$

where $\mathbf{A}=\mathbf{B}+p^{k+1}\cdot\mathbf{C}$. If $\mathbf{A}$ is diagonalisable, we recover the second type of congruence of Atkin and Swinnerton-Dyer.

5.9. *End of proofs.* b) We now make the necessary modifications for the general case of 5.4 and 5.6. Choose N and X as in part b) of the proof of 5.2.

For 5.4 view f as an element of $S_{k+2}(X,\mathbf{Z}_p)^{G_N}$. Its t-expansion is therefore

$$\sum_{n\geqq 1} b(n/N)\cdot t^n.$$

Define $\delta_p\in 1+p\,\mathbf{Z}_p$ by $\delta_p^\mu=D^{p-1}$; thus $\gamma_p=\delta_p^N$. Since $H_p(F)$ annihilates $L_k(X,\mathbf{Z}_p)^{G_N}$, we have in ${}^{(p)}L_k^\infty(X,\mathbf{Z}_p)$

$$H_p(F)\big(\sum_{n\geqq 1} b(n/N)\cdot t^N\big)\equiv 0\qquad(\operatorname{mod}\operatorname{Im}(p\,\partial)^{k+1})$$

and as in a)

$$\sum_{r=0}^{2d} A_r\cdot\gamma_p^{n(p^d-p^{d-r})/(p-1)}\cdot b(n\,p^{d-r}/N)\in(n\,p)^{k+1}\,\mathbf{Z}_p.$$

Writing $N\,n$ for n gives the desired result.

For 5.6, we have

$$\begin{aligned}F(S_{k+2}(X,\mathbf{Z}_p)^{G_N})&\subseteq p^{k+1}\cdot L_k(X,\mathbf{Z}_p)\cap L_k(X,\mathbf{Z}_p)^{G_N}\\&=p^{k+1}\cdot L_k(X,\mathbf{Z}_p)^{G_N}\end{aligned}$$

and so we may write

$$L_k(X,\mathbf{Z}_p)^{G_N}=S_{k+2}(X,\mathbf{Z}_p)^{G_N}\oplus U$$

and

$${}^{(p)}L_k(X,\mathbf{Z}_p)^{G_N}=S_{k+2}(X,\mathbf{Z}_p)^{G_N}\oplus p^{k+1}\cdot U.$$

As above, the image of $p^{k+1}\,U$ in ${}^{(p)}L_k^\infty(X,\mathbf{Z}_p)$ is trivial; the rest of the proof then goes through just as for the general case of 5.4.

5.10. *Remarks.* i) We indicate, without proof, the generalisations of 5.4 and 5.6 to primes $p\nmid M$ with $2<p\leqq k+1$. (See also §§ 7.3, 7.4 of [S].)

As explained in 2.8.ii), the complex $\Omega^\cdot(\mathscr{E}_k)$ must be replaced by $\Omega^\cdot(\Gamma^k(\mathscr{E};\omega))$. Denote by J_k the $\mathbf{H}^1$ of this complex; it replaces L_k. The exact sequence

$$0\to S_{k+2}\to J_k\xrightarrow{\upsilon} S_{k+2}{}^{\vee}\to 0$$

holds without the hypothesis that $k!$ be invertible. ${}^{(p)}L_k$ is replaced by the submodule

$$ {}^{(p)}J_k(X, \mathbf{Z}_p) = \upsilon^{-1}(p^{\langle k+1 \rangle} \cdot S_{k+2}(X, \mathbf{Z}_p)^\vee) $$

where

$$ \langle i \rangle = \inf \left\{ \operatorname{ord}_p \left(\frac{p^j}{j!} \right) :\ j \geqq i \right\} $$

and ${}^{(p)}L_k^\infty$ by

$$ {}^{(p)}J_k^\infty(X, \mathbf{Z}_p) = \frac{t \cdot \mathbf{Z}_p[\![t]\!]}{p^{\langle k+1 \rangle} \cdot \partial^{k+1}(\mathbf{Z}_p[\![t]\!])}. $$

Theorems 5.4 and 5.4 then hold under the hypothesis $p \nmid 2M$, with

$$ (k+1)(\operatorname{ord}_p n + 1) \quad \text{replaced by} \quad (k+1)\operatorname{ord}_p n + \langle k+1 \rangle. $$

ii) We may apply these methods to give congruence properties of holomorphic forms which are not necessarily cusp forms. To simplify matters, assume that Γ is contained in $\pm\Gamma(N)$ for some $N \geqq 3$, and let s denote the number of cusps of Γ. The notation being as in § 4, write

$$ \mathscr{W}_k^! = R^1 c_*(j'_! h'^* \mathscr{F}_k) $$

so that

$$ \mathscr{W}_k^!(\bar{\mathbf{Q}}) = H_c^1(Y \otimes \bar{\mathbf{Q}}, h'^* \mathscr{F}_k) $$

and let $H_p^!(T)$ be the characteristic polynomial of Frob_p on $\mathscr{W}_k^!(\bar{\mathbf{Q}})$. Then

$$ H_p^!(T) = H_p(T) \cdot G_p(T) $$

where $H_p(T)$ is as above, and $G_p(T)$ is a polynomial of degree s, all of whose zeroes are roots of unity. (Indeed, $G_p(T)^{-1}$ is the zeta function of the zero-dimensional variety $Z/\mathbf{F}_p$.) Write

$$ H_p^!(T) = \sum_{r=0}^{2d+s} A_r^! \, T^{2d+s-r} $$

and let $M_{k+2}(\Gamma)$ denote the space of holomorphic modular forms of weight $k+2$ on Γ, for an even integer $k > 0$.

Theorem. *$M_{k+2}(\Gamma)$ is spanned by forms whose Fourier expansions are of the form*

$$ \tilde{f} = \sum_{n \geqq 0} a(n) \cdot \exp(2\pi i n \tau/\mu), \qquad \gamma_\infty^{-n} \cdot a(n) \in \mathbf{Z}[1/M]; $$

moreover, for any such form f, every $n \geqq 1$ and every prime p with $p \nmid 2M$, $p > k+1$,

$$ \operatorname{ord}_p(a(n p^{d+s}) + A_1^! \, a(n p^{d+s-1}) + \ldots + A_{2d+s}^! \, a(n/p^d)) \geqq (\operatorname{ord}_p n + 1)(k+1). $$

The proof is analogous to 5.4; we replace ${}^{(p)}L_k$, ${}^{(p)}L_k^\infty$ by the groups ${}^{(p)}T_k$, ${}^{(p)}T_k^\infty$ discussed in 2.13.iii). There is an operator F on these groups, and on

$$ {}^{(p)}T_k^\infty(X, \mathbf{Z}_p) = \frac{\mathbf{Z}_p[\![t]\!]}{(p\,\partial)^{k+1}(\mathbf{Z}_p[\![t]\!])} $$

it is given by the explicit formula (3.3.2). In the l-adic representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$

$$\mathcal{W}_k^*(\bar{\mathbf{Q}}) := H^1(Y\otimes\bar{\mathbf{Q}}, h'^*\mathcal{F}_k) = \mathrm{Hom}(\mathcal{W}_k^!(\bar{\mathbf{Q}}), \mathbf{Q}_l(-k-1))$$

the geometric Frobenius element Frob_p has characteristic polynomial

$$H_p^*(T) = H_p(T)\cdot p^{(k+1)s} G_p(T/p^{k+1}). \tag{5.10.1}$$

The trace formulae show that $H_p^*(F)$ annihilates ${}^{(p)}T_k(X, \mathbf{Z}_p)$, and the same local calculation gives the congruences. (The passage from H_p^* to $H_p^!$ occurs in the course of this calculation; compare (5.7.1) above.)

There is also an analogue of 5.6; in addition, the "ordinary" case gives rise to a p-adic splitting, analogous to the theory of Eisenstein series, as follows. There is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S_{k+2}(X, \mathbf{Q}_p) \arrow[r] \arrow[d, hook] & L_k(X, \mathbf{Q}_p) \arrow[r] \arrow[d, hook] & S_{k+2}(X, \mathbf{Q}_p)^\vee \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{k+2}(X, \mathbf{Q}_p) \arrow[r] & T_k(X, \mathbf{Q}_p) \arrow[r] & S_{k+2}(X, \mathbf{Q}_p)^\vee \arrow[r] & 0.
\end{tikzcd}$$

If $H_p(T)$ is ordinary, the unit root subspace

$$U\otimes\mathbf{Q}_p \subset L_k(X, \mathbf{Q}_p)$$

splits both rows of this diagram, by (5.7.2). The eigenvalues of F on $L_k(X, \mathbf{Q}_p)$ and $T_k(X, \mathbf{Q}_p)/L_k(X, \mathbf{Q}_p)$ have different archimedean absolute values, whence the action of F splits the second vertical arrow (the "weight" splitting). The first vertical arrow therefore also has a canonical splitting:

$$M_{k+2}(X, \mathbf{Q}_p) = S_{k+2}(X, \mathbf{Q}_p) \oplus \mathrm{Eis}_{k+2,p}(X)$$

where

$$\mathrm{Eis}_{k+2,p}(X) := \{x \in M_{k+2}(X, \mathbf{Q}_p) \colon G_p(F/p^{k+1})(x) \in U\otimes\mathbf{Q}_p\}.$$

Of course, when Γ is a congruence subgroup $\mathrm{Eis}_{k+2,p}(X)$ is simply the space spanned by the Eisenstein series, since F commutes with the Hecke operators.

iii) It is clear that the principles of the proofs will apply equally well to forms of odd weight $\geqq 3$ on a subgroup $\Gamma \leqq SL_2(\mathbf{Z})$ of finite index, not containing -1. Indeed, if $\Gamma \leqq \pm\Gamma(N)$ for some $N \geqq 3$, and if there is a model $V/\mathbf{Q}$ for $\pm\Gamma\backslash\mathfrak{H}^*$ as in (5.2.2), (5.2.3), then the arguments above carry through without change. The general case is slightly more complicated; to use the method of intersecting Γ with $\Gamma(N)$ for suitable N, the following hypothesis seems to be required:

For some integer $N \geqq 3$ for which $\Gamma\cdot\Gamma(N) = SL_2(\mathbf{Z})$, there is a model $V/\mathbf{Q}$ for the Riemann surface $\pm(\Gamma\cap\Gamma(N))\backslash\mathfrak{H}^*$ as in 5.1, such that the j-morphism $\pi\colon V \to \mathbf{P}^1_{\mathbf{Q}}$ factors through $X(N)_{\mathbf{Q}}$, and such that there is an action of G_N on V, which is compatible with its action on $X(N)_{\mathbf{Q}}$ and with the action of $G_N(\mathbf{C}) \simeq \Gamma/\Gamma\cap\Gamma(N)$ on $V(\mathbf{C})$.

(Note that taking the fibre product of $X(N)$ with a model for $\pm\Gamma\backslash\mathfrak{H}^*$ will not give a suitable V, but rather a model for $(\pm\Gamma\cap\pm\Gamma(N))\backslash\mathfrak{H}^*$.)

It would be desirable to have an alternative hypothesis which did not involve the auxiliary integer N.

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