

# Galois Theory (Michaelmas 2013): Zorn's lemma

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In order to prove the existence of the algebraic closure of an arbitrary field, it is necessary to use an axiom of set theory known as *Zorn's lemma*. It is equivalent to the Axiom of Choice (but I won't prove the equivalence here: see the part II course "Logic and Set Theory", Halmos's *Nave Set Theory* or any other book on set theory for a proof and discussion). Some believe that one should avoid the Axiom of Choice wherever possible, as it is less intuitive than the other axioms of set theory. However a lot of algebra (not to say analysis) would be very awkward without it (see Theorem A below for one reason why). If one is really concerned about its validity, it is worth pointing out that one can often avoid using Zorn's Lemma, at the expense of some notational complexity (for example, instead of the algebraic closure of a field one can often make do with the splitting field of a sufficiently large finite set of polynomials).

The material below (other than the statement of Zorn's Lemma and proof of Theorem A) is not examinable.

## Partial orders and Zorn's lemma

Let  $S$  be a set. A relation  $\leqslant$  on  $S$  is said to be a *partial order* if it satisfies:

- (i) For all  $x \in S$ ,  $x \leqslant x$ ;
- (ii) For all  $x, y, z \in S$ , if  $x \leqslant y$  and  $y \leqslant z$  then  $x \leqslant z$ ;
- (iii) For all  $x, y \in S$ , if  $x \leqslant y$  and  $y \leqslant x$  then  $x = y$ .

$S$  is said to be *totally ordered* by  $\leqslant$  if moreover:

- (iv) For all  $x, y \in S$ , either  $x \leqslant y$  or  $y \leqslant x$ .

A *chain* in a partially ordered set  $(S, \leqslant)$  is a subset  $T \subset S$  which is totally ordered by  $\leqslant$ . If  $T \subset S$  is a chain then so is any subset of  $T$ .

### Examples:

- (a)  $\mathbb{R}$  is a totally ordered set (with the usual order relation).
- (b) Let  $S = \{x \in \mathbb{Z} \mid x > 1\}$  ordered by reverse divisibility:

$$x \preccurlyeq y \iff \frac{x}{y} \in \mathbb{Z}.$$

Then  $(S, \preccurlyeq)$  is a partially ordered set. Let  $m > 1$  and  $T = \{m^i \mid i > 1\}$ . Then  $T$  is a chain in  $S$ . So is the subset  $\{n! \mid n > 1\}$ .

- (c) Let  $X$  be any set,  $S$  the set of all subsets of  $X$  with inclusion as the order relation. Then  $S$  is a partially ordered set.

Let  $(S, \leqslant)$  be a partially ordered set, and  $T$  any subset of  $S$ . An *upper bound* for  $T$  is an element  $z \in S$  such that  $x \leqslant z$  for all  $x \in T$ . (We don't require that  $z \in T$ .) An element  $y \in S$  is said to be *maximal* if for any  $x \in S$ ,  $y \leqslant x$  iff  $x = y$ .

If  $S$  is totally ordered, then it can have at most one maximal element (easy). A general partially ordered set can have many maximal elements. In the above examples:

- (a) In  $\mathbb{R}$  an upper bound for a subset is an upper bound in the usual sense. There are no maximal elements.
- (b) In  $S = \{x \in \mathbb{Z} \mid x > 1\}$  an element  $x \in S$  is a maximal element iff it is prime. Every chain has an upper upper bound (take the element which is smallest for the usual ordering on  $\mathbb{N}$ ).

**Zorn's Lemma.** *Let  $S$  be a nonempty partially ordered set. Assume that every chain in  $S$  has an upper bound. Then  $S$  has a maximal element.*

As an example of the uses of Zorn's lemma, we prove:

**Theorem A.** *Let  $R$  be a ring (nonzero, with unit element). Then  $R$  has a maximal ideal.*

*Proof.* Let  $S$  be the set of all proper (i.e. different from  $R$  itself) ideals of  $R$ , ordered by inclusion. This is clearly a partial ordering on  $S$ . Since  $R$  is nonzero,  $\{0\} \in S$  and so  $S$  is nonempty. The maximal elements of  $S$  are precisely the maximal ideals of  $R$ . We need to check the hypothesis of Zorn's lemma. Let  $T \subset S$  be a chain. Define  $J = \bigcup_{I \in T} I$  – we claim that  $J$  is an upper bound for  $T$ . Evidently for every  $I \in T$  we have  $I \subset J$ , so the only thing to check is that  $J \in S$ . As  $J$  is a union of ideals, it is an ideal of  $R$ . Moreover it is a proper ideal, for if not then  $1 \in J$  which is true iff  $1 \in I$  for some  $I \in T$ , which is impossible as  $I$  is a proper ideal. Therefore  $J \in S$  and so  $J$  is an upper bound for  $T$ . By Zorn's lemma,  $S$  has maximal elements, hence  $R$  has maximal ideals.  $\square$

**Corollary.** *Let  $R$  be a ring,  $I \subsetneq R$  a proper ideal. Then there is a maximal ideal of  $R$  containing  $I$ .*

*Proof.* Apply Theorem A to  $R/I$ .  $\square$

Zorn's lemma is equivalent to two other axioms of Set Theory: the first of these is:

**The Axiom of Choice.** *Let  $X_i$  ( $i \in I$ ) be a collection of sets, indexed by a set  $I$ . If each  $X_i$  is nonempty then so is the Cartesian product  $\prod_{i \in I} X_i$ .*

The second requires a further definition. A totally ordered set is said to be *well-ordered* if every non-empty subset contains a least element. For example:

- Every totally ordered finite set is well-ordered.
- The set  $\mathbb{N}$  with its usual ordering is well-ordered.
- $\mathbb{Z}$ , with its usual ordering, is *not* well-ordered (since the set itself doesn't have a least element). Likewise, neither  $\mathbb{Q}$  nor  $\mathbb{R}$ , with their usual ordering, are well-ordered.

**The Well-Ordering Principle.** *Every set can be well-ordered. In other words, for any set  $S$  there exists a total ordering on  $S$  for which it is well-ordered.*

Since  $\mathbb{N}$  is well-ordered, any countable set  $S$  can be well-ordered: there exists an injective map  $f: S \rightarrow \mathbb{N}$ , and just take the ordering on  $S$  to be given by  $x \leq y$  iff  $f(x) \leq f(y)$ . So although  $\mathbb{Q}$  is not well-ordered with respect to the usual ordering, it still satisfies the conclusions of Zorn's lemma, for some ordering (which can be written down explicitly). However for the uncountable set  $\mathbb{R}$  it is not possible to write down an explicit well-ordering — but Zorn's Lemma says that one does indeed exist.

Here is another application of Zorn's Lemma.

**Theorem B.** *Every vector space has a basis.*

*Proof.* (Sketch) Let  $V$  be a vector space. If  $V = \{0\}$  there is nothing to prove, so we may assume  $V$  is nonzero. Let  $S$  be the set whose elements are the linearly independent subsets of  $V$ , ordered by inclusion. Then  $S$  is a nonempty partially ordered set. A basis of  $V$  is nothing other than a maximal element of  $S$ . One checks that if  $T \subset S$  is a chain, then  $\bigcup_{I \in T} I$  is also a linearly independent subset of  $V$ , hence is an upper bound for  $T$ . By Zorn's lemma,  $S$  has maximal elements, so there exists a maximal linearly independent subset of  $V$ , i.e. a basis for  $V$ .  $\square$