Part III: Differential geometry (Michaelmas 2010)

Some facts from multilinear algebra

Let V be a vector space over $\mathbb{R}$. Assume, for simplicity, that V has finite dimension n say.

1. If W is another real finite-dimensional vector space then the **tensor product** $V \otimes W$ may be defined as the real vector space consisting of all formal linear combinations of elements $v \otimes w$ (for $v \in V$, $w \in W$), with the relations

$$(av) \otimes w = v \otimes (aw) = a(v \otimes w),$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \text{ and } v \otimes (w_1 + w_2) = v \otimes w_1 + u \otimes w_2,$$

for all $a \in \mathbb{R}$; $v, v_1, v_2 \in V$; $w, w_1, w_2 \in W$. If $\{v_i\}$ and $\{w_j\}$ are bases of V, W then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$ and thus $\dim(V \otimes W) = \dim V \dim W$. For example, $\mathbb{R}^m \otimes \mathbb{R}^n \cong \mathbb{R}^{mn}$. Consider the **dual space** V^* of linear functions $V \to \mathbb{R}$, and similarly W^*, $(V \otimes W)^*$. Assigning to each $v^* \otimes w^* \in V^* \otimes W^*$ the linear function on $V \otimes W$ determined by

$$v_i \otimes w_j \in V \otimes W \mapsto v^*(v_i)\, w^*(w_j) \in \mathbb{R}$$

defines a natural isomorphism of vector spaces $V^* \otimes W^* \cong (V \otimes W)^*$.
Assigning to a bilinear form ψ on $V \times W$ the linear function

$$v \otimes w \in V \otimes W \to \psi(v, w) \in \mathbb{R}$$

defines a natural isomorphism between the vector space of all bilinear forms on $V \times W$ and the dual space $(V \otimes W)^*$. Recall from linear algebra that the space of bilinear forms on $V \times U$ may be naturally identified with the space $L(V, U^*)$ of linear maps $V \to U^*$. Putting $U^* = W$ and noting the above relations, one obtains a natural linear isomorphism

$$L(V, W) \cong V^* \otimes W.$$

In the special case $W = \mathbb{R}$ this recovers the definition of dual vector space V^*.

2. Again let $v_1, \ldots, v_n$ be a basis of V. Then the dual space V^* of linear functions $V \to \mathbb{R}$ can be given the **dual basis** $\lambda_1, \ldots, \lambda_n$, defined by the property

$$\lambda_j(v_j) = \delta_{ij}.$$

Here δ_{ij} is the 'Kronecker delta', $\delta_{ij} = 1$ if $i = j$ and is 0 otherwise. The p-th **exterior power** $\Lambda^p V^*$ of V^* ($p \geq 0$) is the vector space of all the functions $h : V \times \ldots \times V \to \mathbb{R}$, such that h is
(1) **multilinear**: $h(u_1, \ldots, au_i + bu_i', \ldots, u_p) = ah(u_1, \ldots, u_i, \ldots, u_p) + bh(u_1, \ldots, u_i', \ldots, u_p)$; (i.e. linear in each argument) and
(2) **antisymmetric**: $\forall i < j \in \{1, 2, \ldots, n\}$, swapping the i^{th} and j^{th} vector changes the sign

$$h(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{j-1}, u_j, u_{j+1}, \ldots, u_n) = - h(u_1, \ldots, u_{i-1}, u_j, u_{i+1}, \ldots, u_{j-1}, u_i, u_{j+1}, \ldots, u_n).$$

It follows that for $1 \leq p \leq n$, $\dim \Lambda^p V^* = \binom{n}{p}$, a basis may be given by $\{\lambda_{i_1} \wedge \ldots \wedge \lambda_{i_p} : \lambda_{i_j} \in V^*,\ 1 \leq i_1 < \ldots < i_p \leq n\}$. Also $\Lambda^p V^* = \{0\}$ when $p > n$. One formally defines $\Lambda^0 V^* = \mathbb{R}$.

The **exterior product** (or wedge product) is a bilinear map

$$(\lambda, \mu) \in \Lambda^p V^* \times \Lambda^q V^* \to \lambda \wedge \mu \in \Lambda^{p+q} V^*.$$

determined for the basis vectors $\lambda_i \in V^*$, and inductively on p, q, by

$$(\lambda_{i_1} \wedge \ldots \wedge \lambda_{i_p}) \wedge (\lambda_{i_{p+1}} \wedge \ldots \wedge \lambda_{i_{p+q}})(u_1, \ldots, u_{p+q}) = \det(\lambda_{i_j}(u_k)), \qquad (u_k \in V, \;\; j, k = 1, \ldots, p+q).$$

(and extended by linearity). It follows, in particular, that the $\wedge$ is associative $(\lambda \wedge \mu) \wedge \nu = \lambda \wedge (\mu \wedge \nu)$, and $\mu \wedge \lambda = (-1)^{pq} \lambda \wedge \mu \quad (\lambda \in \Lambda^p V^*, \;\; \mu \in \Lambda^q V^*)$.

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