TORIC GEOMETRY, SHEET II: MICHAELMAS 2019

If you would like feedback on your work, you may turn in any two problems into my pigeon hole in the CMS before lunch on November 18.

1. Construct a fan Σ and a proper map to $\mathbb{R}_{\geq 0}$ such that for the associated toric morphism $X \to \mathbb{A}^1$ is proper, every nonzero fiber is isomorphic to $\mathbb{P}^1$, and the zero fiber has exactly r irreducible components.
2. Let X be a product of projective spaces and let dim $X = n$. The dense torus $(\mathbb{C}^\star)^n$ contains a compact real torus $(S^1)^n$ where the norm of each complex number to be 1. Describe the quotient space $X^{\text{an}}/(S^1)^n$.
3. Consider the variety X obtained by taking $\mathbb{P}^n$ and blowing up a single closed dimension k coordinate plane. Carefully describe the fan for this variety (the answer will certainly depend on k). Identify all the torus orbit closures of X.
4. On $(\mathbb{C}^\star)^n$, there is a map sending a tuple of nonzero complex numbers to the tuple of their reciprocals, and is known as the *Cremona transform*. Describe the self-map on the cocharacter lattice that induces this map. Does this map extend to the compactification $\mathbb{P}^n$?
5. Let X be the blowup of $\mathbb{P}^2$ at its three torus fixed points and let $\pi : X \to \mathbb{P}^2$ be the blowup morphism. Prove that the coordinatewise reciprocal map defined in the previous question extends to an endomorphism $\phi : X \to X$. Let ℓ be a line in $\mathbb{P}^2$ that does not pass through the coordinate points, and therefore is isomorphic to a unique subvariety in X. Give an explicit description of the subvariety of $\mathbb{P}^2$ obtained as $\pi \circ \phi \circ \pi^{-1}(\ell)$.
6. Construct a toric variety X with dense torus $T \cong (\mathbb{C}^\star)^n$ with the following two properties: (1) the Cremona transform on T extends to a regular (i.e. well-defined) self-map on X, and (2) admits a toric birational morphism $X \to \mathbb{P}^n$.
7. The Picard group[1] of a smooth toric variety is always a finitely generated abelian group and the *Picard rank* of a toric variety X is the rank of its Picard group. Give examples to show that the Picard rank of toric surfaces can be unbounded, i.e. for any integer N there is a toric surface with Picard rank larger than N.
8. Prove or give a counterexample: the divisor class group of a toric variety is always torsion free.
9. Let σ be the 3-dimensional cone obtained as a cone over a square. Identify[2] the associated toric variety with the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Give a toric resolution of singularities

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[1]The Picard group was defined in lecture as the group of Cartier divisors up to rational equivalence. You may take as a given that it coincides with the class group for smooth toric varieties.

[2]You may need to look at the Wikipedia pages for Segre embedding and for affine cone.

$\pi : X \to U_\sigma$ with the additional constraint that the morphism is an isomorphism in codimension 1. Practically, this is requiring that π is a bijection on torus orbits of dimension 2 and 3.

10. Prove or give a counterexample: For toric morphisms of toric varieties $X_1 \to Z$ and $X_2 \to Z$ the fiber product of schemes $X_1 \times_Z X_2$ is always a toric variety.
11. Given an arbitrary fan Σ in $N_\mathbb{R}$ prove that there exists a subdivision $\widetilde{\Sigma}$ of Σ such that every cone of $\widetilde{\Sigma}$ is generated by a subset of a vector space basis for $N_\mathbb{R}$ (i.e. is simplicial)[3]. ($\star$) Prove the ultimate statement: there is a further refinement of $\widetilde{\Sigma}$ where every cone is generated by a lattice basis.
12. Give a toric morphism of toric varieties $f : X \to Y$ and a point $y \in Y$ such that the scheme theoretic fiber $X \times_Y \{y\}$ is non-reduced. ($\star$) Can you give a combinatorial condition that guarantees that all scheme theoretic fibers of f are reduced?

[3] A more pretentious way of saying this is that X_Σ has a toric resolution of singularities by a smooth orbifold, or even more pretentiously, a smooth Deligne–Mumford stack.