Ailsa Keating

Symplectic topology: Example Sheet 3 of 4

1. If (M^{2n}, ω) is a symplectic manifold and $\pi : \tilde{M} \to M$ is the blow-up of M at a point $p \in M$ with exceptional divisor $E = \pi^{-1}(p)$, show $c_1(\tilde{M}) = \pi^* c_1(M) - (n-1)\mathrm{PD}(E)$. Hence, or otherwise, show $\sigma + e$, the sum of signature and Euler characteristic, is invariant under blowing up/down four-manifolds. Compute the first Chern class of the elliptic surface $E(1)$ given by blowing up $\mathbb{P}^2$ the base-points of a pencil of cubics.

2. Let $E(1)$ be an elliptic surface given by blowing up $\mathbb{P}^2$ on base-points of a pencil of cubics. Let $T \subset E(1)$ be a smooth fibre. Show that $\pi_1(E(1) \backslash T) = 0$.

3. (Wait until 26/02, where a more general case will be covered:) Construct a closed symplectic four-manifold with fundamental group $\mathbb{F}_3$, the free group on 3 generators.

4. Let $\mathbb{F}_1$ denote the first Hirzebruch surface, given by blowing up $\mathbb{P}^2$ at a point p. Let $H \subset \mathbb{F}_1$ denote a line in $\mathbb{F}_1$, i.e. the pullback of a line in $\mathbb{P}^2$ *not* passing through the point p, and $E \subset \mathbb{F}_1$ the exceptional curve. Give two proofs that there is no symplectic form on $\mathbb{F}_1$ giving H and E the same area: (i) using the cohomology ring and (ii) using symplectic fibre summation.

5. (a) Find a formula for the Euler characteristic of the total space of a Lefschetz pencil in terms of the genus of a smooth fibre, the number of base points, and the number of critical points.

   (b) Equip the four-torus T^4 with its standard symplectic structure. How many base-points can a Lefschetz fibration of genus g curves on T^4 have?

6. Let $H : M \to \mathbb{R}$ be a Hamiltonian function with level set $\Sigma = H^{-1}(t)$. If $L \subset M$ is a Lagrangian submanifold lying entirely inside Σ, show the Hamiltonian flow of H preserves L.

7. (Revisited from ES2.) (a) Show that the complement of the diagonal in $(\mathbb{P}^1 \times \mathbb{P}^1, \omega \oplus \omega)$ is diffeomorphic to T^*S^2. What can you say at the symplectic level?

   (b) Show that $\mathbb{P}^2$ is biholomorphic to the 2-fold symmetric product of $\mathbb{P}^1$, i.e. the quotient of $\mathbb{P}^1 \times \mathbb{P}^1$ by the involution which exchanges the two factors. [Hint: Look for functions invariant under the S_2 action.] What do the diagonal and anti-diagonal get mapped to? (Harder:) How are the symplectic forms on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ related in this set-up?