Easter Term 2019 Ailsa Keating

Symplectic topology: Revision Sheet

Examination rubrik: attempt 3 out of 4 questions. Extra questions included here for practice purposes.

1. What is a Lefschetz pencil on a smooth oriented 4-manifold M?

Let $f : M \dashrightarrow S^2$ be a Lefschetz pencil for which the critical points all map to distinct critical values. Let $p \in S^2$ be a regular value and $\gamma : [0,1] \to S^2$ a smooth path with $\gamma(0) = p$, $\gamma(1) \in \mathrm{Crit}(f)$ and $\gamma|_{[0,1)}$ contained in the regular values of f. Prove that γ defines a smooth "vanishing cycle", a sphere $L_\gamma \subset f^{-1}(p)$.

Now let the Lefschetz pencil $M^4 \dashrightarrow S^2$ be on a smooth four-manifold with fibres of genus $g > 0$. Describe without proof the monodromy of the fibration around a path encircling one critical value. Deduce there is no choice of embedded paths $\{\gamma_j \mid j \in \mathrm{Crit}(f)\}$ from p to the distinct critical values for which the associated circles L_γ are all in the same nonzero homology class.

2. What does it mean to blow up a symplectic manifold at a point $p \in X$? Prove that the blow-up of X at p carries a symplectic structure. Is this well-defined up to symplectomorphism?

State without proof a formula for the first Chern class of the blow-up.

Equip the four-torus T^4 with its standard symplectic structure. Show that a Lefschetz pencil of genus g curves on T^4 has exactly $2g - 2$ base points.

3. What does it mean for an almost complex structure J to be compatible with a symplectic form? Prove that every symplectic manifold (M, ω) has a connected non-empty space of compatible almost complex structures.

Assume J is compatible with ω. What does it mean for a map $u : \mathbb{P}^1 \to M$ to be J-holomorphic? Explain what it means for J to be regular for u.

Now suppose that J is regular for all J-holomorphic curves $u : \mathbb{P}^1 \to M$, and $\dim_{\mathbb{R}} M = 4$. State a theorem giving the dimension of the space of J-holomorphic spheres representing a class $A \in H_2(M;\mathbb{Z})$. Deduce that if $[A] \cdot [A] \leq -2$ then the space of J-holomorphic spheres in class A is empty.

4. What is a symplectic capacity on a symplectic manifold (X, ω)?

Assuming the existence of a symplectic capacity c, prove that $\mathrm{Symp}(X)$ is C^0-closed in $\mathrm{Diff}(X)$, the group of all diffeomorphisms of X.

Now let c be a capacity on a subset of $\mathbb{R}^{2n}$. Let $U \subset \mathbb{R}^{2n}$ be an open non-empty bounded subset and $W \subset \mathbb{R}^{2n}$ a codimensions two linear subspace. Write $U + W = \{u + w \,|\, u \in U, w \in W\}$. Prove $0 < c(U + W) < \infty$ if $W^{\perp}$ is not isotropic. [Hint: to show finiteness, embed $U + W$ in something standard.]

Additional questions.

5. State and prove Darboux's theorem on the local structure of a symplectic manifold. State without proof Weinstein's neighbourhood theorem for closed Lagrangian submanifolds of a symplectic manifold.

Let M be a simply connected symplectic manifold. Suppose there are symplectomorphisms $\phi_k : M \to M$ for which
(i) each ϕ_k has a unique fixed point in M;
(ii) the $\phi_k \to \mathrm{id}$ in the C^0-topology on the group $Symp(M)$ of symplectomorphisms of M.
Prove M is not closed.

6. What is a Kähler manifold?

Considering the two-form $\partial\overline{\partial}\log(|z|^2 + 1)$ on $\mathbb{C}^{n+1}$, or otherwise, prove that a closed complex submanifold of $\mathbb{CP}^n$ is Kähler.

Which of the following statements are true and which false? Justify your answers briefly.
(i) The product $S^2 \times T^2$ is Kähler.
(ii) The connect sum $(S^2 \times T^2)\#(S^1 \times S^3)$ is symplectic.
(iii) The connect sum $(S^2 \times T^2)\#(S^1 \times S^3)\#(S^1 \times S^3)$ is Kähler.