Symplectic topology: Example Sheet 3 of 4

1. If (M^{2n}, ω) is a symplectic manifold and $\pi : \tilde{M} \to M$ is the blow-up of M at a point $p \in M$ with exceptional divisor $E = \pi^{-1}(p)$, show $c_1(\tilde{M}) = \pi^* c_1(M) - (n-1)\mathrm{PD}(E)$. Hence, or otherwise, show $\sigma + e$, the sum of signature and Euler characteristic, is invariant under blowing up/down four-manifolds. Compute the first Chern class of the elliptic surface $E(1)$ given by blowing up $\mathbb{P}^2$ the base-points of a pencil of cubics.

2. Let $E(1)$ be an elliptic surface given by blowing up $\mathbb{P}^2$ the base-points of a pencil of cubics. Let $T \subset E(1)$ be a smooth fibre. Show that $\pi_1(E(1)\backslash T) = 0$.

3. (Review from lectures:) Construct a closed symplectic four-manifold with fundamental group $\mathbb{F}_3$, the free group on 3 generators.

4. Show that the symplectic fibre sum of the four-torus T^4 with $T^2 \times S^2$, along a fixed pair of symplectic tori C_1 and C_2 of square zero, can yield topologically distinct symplectic manifolds, depending on the framings of the normal bundles of the C_i used in the construction.

5. Let $\mathbb{F}_1$ denote the first Hirzebruch surface, given by blowing up $\mathbb{P}^2$ at a point p. Let $H \subset \mathbb{F}_1$ denote a line in $\mathbb{F}_1$, i.e. the pullback of a line in $\mathbb{P}^2$ *not* passing through the point p, and $E \subset \mathbb{F}_1$ the exceptional curve. Give two proofs that there is no symplectic form on $\mathbb{F}_1$ giving H and E the same area: (i) using the cohomology ring and (ii) using symplectic fibre summation.

6. Find a formula for the Euler characteristic of the total space of a Lefschetz pencil in terms of the genus of a smooth fibre, the number of base points, and the number of critical points.

7. Find a symplectic four-manifold which admits a (non-product) Lefschetz pencil of genus 2, but not of genus 1, curves.

8. Let $H : M \to \mathbb{R}$ be a Hamiltonian function with level set $\Sigma = H^{-1}(t)$. If $L \subset M$ is a Lagrangian submanifold lying entirely inside Σ, show the Hamiltonian flow preserves L.

9. Show that a smooth J-holomorphic submanifold in a symplectic manifold is a symplectic submanifold.

10. (a) Find a complex algebraic surface X and a homology class $A \in H_2(X; \mathbb{Z})$ so that there are holomorphic curves in the class A, but none of these curves are connected.
(b) Find a complex algebraic surface X and a homology class $A \in H_2(X; \mathbb{Z})$ so that there are singular (nodal) holomorphic curves in the class A, but no smooth holomorphic curves.

11. The Nijenhuis tensor of an almost complex structure J associates to vector fields X and Y

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

Show that $(T^{1,0}M, J)$ is closed under Lie bracket iff $N_J \equiv 0$. Deduce this is true for any J on a 2-dimensional surface, so all almost complex structures on surfaces are integrable in this sense.