Symplectic topology: Example Sheet 2 of 4

1. On fibrations:

   (i) Show the torus T^4 admits fibrations over T^2 with symplectic fibres, and fibrations with Lagrangian fibres. Is there a Lagrangian fibration of $S^2 \times T^2$?

   (ii) Suppose $f : X^4 \to C^2$ is a closed oriented four-manifold which fibres smoothly over a closed oriented surface C. Is X necessarily symplectic? [Harder: Can you give a sufficient condition on the fibers of f for X to be symplectic?]

2. Let $\mathcal{J}_{2n}$ denote the set of almost complex structures on $\mathbb{R}^{2n}$. Show $GL_{2n}(\mathbb{R})$ acts transitively on constant coefficient almost complex structures on the vector space $\mathbb{R}^{2n}$ (i.e. on matrices of square $-I$). Deduce $\mathcal{J}_4 \simeq S^2 \amalg S^2$, the disjoint union of two 2-spheres.

3. On the projective plane:

   (i) What is the genus of a smooth degree d curve in $\mathbb{CP}^2$?

   (ii) Show there is a Lagrangian surface in $\mathbb{CP}^2$ which intersects every elliptic (cubic) curve.

   (iii) Construct a Lagrangian surface in $\mathbb{CP}^2$ which can be displaced off itself by a Hamiltonian isotopy (i.e. by the flow of a Hamiltonian vector field).

4. Let λ_{can} be the canonical one-form in $\Omega^1(T^*M)$. Show that for any one-form $\sigma \in \Omega^1(M)$, thought of as a map $M \to T^*M$, we have that $\sigma^*\lambda_{can} = \sigma$, and that this property characterises λ_{can}.

5. Prove the Poincaré lemma from lectures: let $Q \subset M$ be a closed smooth submanifold, and let $\alpha_1, \alpha_2 \in \Omega^k(M)$ be closed differential forms agreeing on $TM|_Q$; show that there is a form $\beta \in \Omega^{k-1}(U_Q)$, defined on some open nhood U_Q of Q, such that $d\beta = \alpha_1 - \alpha_2$ and $\beta = 0$ on $TM|_Q$.

6. Carefully write down a proof of the neighbourhood theorem for isotropic submanifolds sketched in lectures: suppose that (X, ω) is symplectic, and that $W \subset X$ is an isotropic submanifold; prove that the neighbourhood of W is determined symplectically by the smooth topology of W and the bundle $TW^\perp/TW$.

7. Suppose that (M, ω) contains two Lagrangian submanifolds L_1, L_2 which meet transversally at a point p. Show that there is a Darboux chart centered on p, say $\phi : B^{2n}(\epsilon) \to M$, such that $\phi^{-1}(L_1) = \mathbb{R}^n \cap B^{2n}(\epsilon)$ and $\phi^{-1}(L_2) = i\mathbb{R}^n \cap B^{2n}(\epsilon)$.

8. Show that the quotient of $\mathbb{R}^4$ by the group generated by the four transformations $(x, y, z, t) \mapsto (x+1, y, z, t); (x, y+1, z, t); (x, y, z+1, t); (x+y, y, z, t+1)$ admits a symplectic but not a Kähler structure. [In fact, this manifold also admits a complex structure: why is this not a contradiction?]

9. Let (M, ω) be symplectic and fix a compatible almost complex structure J. Show that smoothly embedded J-holomorphic curves $C \subset M$, i.e. two-dimensional surfaces C with TC preserved by J, minimise symplectic area $\int_C \omega$ amongst embedded surfaces in their homology class.

10. Show the complement of a smooth conic curve in $\mathbb{CP}^2$ contains a Lagrangian $\mathbb{RP}^2$; show the complement of the diagonal in $(\mathbb{P}^1 \times \mathbb{P}^1, \omega \oplus \omega)$ contains a Lagrangian S^2. Are these two statements related?