Mapping class groups
Problem sheet 3

Lent 2021

Questions marked with a * are optional.

1. Let S be hyperbolic, and let Σ be a connected, finite-sheeted covering space of S. Prove that there is a subgroup of finite index $\Gamma \leq \mathrm{Mod}(S)$, a subgroup $\Gamma' \leq \mathrm{Mod}(\Sigma)$ and a short exact sequence

$$1 \to K \to \Gamma' \to \Gamma \to 1\,,$$

   where K is the group of deck transformations of Σ over S.

2. Consider a closed, orientable surface S of genus $g > 1$.

   (i) What is the maximal dimension of a simplex of the complex of curves $C(S)$?

   (ii) Prove that the number of $\mathrm{Mod}(S)$-orbits of maximal simplices is equal to the number of connected, trivalent graphs with $2g-2$ vertices.

   (iii) How many orbits of maximal simplices are there when $g = 2$?

3. If $g > 1$, prove the complex of curves $C(S_g)$ is locally infinite; that is, every vertex of $C(S_g)$ adjoins infinitely many edges.

4. Let S be closed and hyperbolic. For a pair of essential simple closed curves α, β on S, let $d(\alpha, \beta)$ be the number edges in the shortest path in the 1-skeleton of $C(S)$ between the isotopy classes of α and β. Prove that $d(\alpha, \beta) \leq 2i(\alpha, \beta)$ as long as $i(\alpha, \beta) \geq 1$. Is there an inequality in the other direction?

5. Prove the following variant of Lemma 10.6 from lectures. Let X be a path-connected simplicial complex, and let G be a group acting on X by simplicial automorphisms. Suppose that Y is a subcomplex whose G-translates cover X; that is, $GY = X$. Then the set of elements

$$\{g \in G \mid gY \cap Y \neq \varnothing\}$$

generates G.

6. Consider a surface S with $n > 0$ punctures. The *arc graph* $A(S)$ is defined as follows. The vertices are isotopy classes of unoriented, simple, properly embedded arcs in S. Two vertices α, β are joined by an edge if $i(\alpha, \beta) = 0$.

   (a) Describe the arc graph of the once-punctured torus T^2_*. Draw a natural picture of $A(T^2_*)$ in the (compactified) upper half-plane.

   (b) Prove that $A(T^2_*)$ is connected.

7. Define a variant of the curve complex of the torus, $C'(T^2)$, as follows. The vertices are isotopy classes of unoriented, essential, simple closed curves. Vertices represented by curves α, β are joined by an edge if $i(\alpha, \beta) = 1$.

   (a) Prove that $C'(T^2)$ is connected.

   (b) Formulate a similar definition for $C'(S_{0,0,4})$, where $S_{0,0,4}$ is the 4-holed sphere, and prove that your $C'(S_{0,0,4})$ is connected.

8. Prove that $SL_2(\mathbb{Z})$ is generated by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

9. * Let S be a compact hyperbolic surface. A *pants decomposition* $\underline{\alpha}$ of S is a multicurve α on S such that every component of the cut surface S_α is homeomorphic to a pair of pants. The *pants graph* $P(S)$ is defined as follows. The vertices are isotopy classes of pants decompositions of S. Two vertices

$$\underline{\alpha} = \alpha_1 \sqcup \ldots \sqcup \alpha_n, \ \underline{\beta} = \beta_1 \sqcup \ldots \sqcup \beta_n$$

are joined by an edge if, after renumbering:

(a) α_i is isotopic to β_i for all $i > 1$;

(b) if $S_{\alpha_2,\ldots,\alpha_n}$ is a one-holed torus then $i(\alpha_1, \beta_1) = 1$;

(c) if $S_{\alpha_2,\ldots,\alpha_n}$ is a four-holed sphere then $i(\alpha_1, \beta_1) = 2$.

Prove that $P(S)$ is connected for sufficiently complicated surfaces S. [*Hint: Find a natural embedding of* $P(S)$ *into* $C(S)$.] What is the stabiliser of a vertex in the pants graph?