

Mapping class groups

Problem sheet 2

Michaelmas 2019

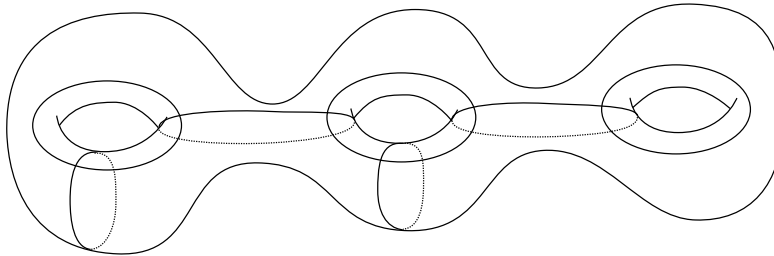
1. What is the maximal cardinality of a pairwise non-isotopic set of essential simple closed curves on a closed surface S ?
2. Exhibit $\phi \in \text{Mod}(S_g)$ of order g .
3. Let S be the closed, orientable surface obtained from a decagon P_{10} by identifying opposite sides in pairs.
 - (a) What is the genus of S ?
 - (b) Let ϕ be the rotation of P_{10} by π . Show that ϕ induces a self-homeomorphism of S . What is its order in $\text{Mod}(S)$?
4. Let γ be an element of a group G . The corresponding *inner automorphism* of G is defined by

$$i_\gamma : g \mapsto \gamma g \gamma^{-1} .$$

Let $\text{Inn}(G)$ denote the image of the map $G \rightarrow \text{Aut}(G)$ sending $\gamma \mapsto i_\gamma$.

- (a) Prove that $\gamma \mapsto i_\gamma$ is a homomorphism and that $\text{Inn}(G) \triangleleft \text{Aut}(G)$. What is the kernel of the homomorphism $G \rightarrow \text{Inn}(G)$?
- (b) The quotient $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ is called the *outer automorphism group* of G . Exhibit a homomorphism $\text{Mod}(S) \rightarrow \text{Out}(\pi_1 S)$.
- (c) Is the homomorphism $\text{Mod}(S) \rightarrow \text{Out}(\pi_1 S)$ surjective?

5. Let G be any finite group.
 - (a) Show that there is a surjection $\pi_1 S_g \rightarrow G$, for some g .
 - (b) Show that G is a subgroup of $\text{Mod}(S_h)$, for some h .
6. Let S be closed and hyperbolic, and let $\phi : S \rightarrow S$ be an isometry isotopic to the identity. Show that ϕ is equal to the identity.
7. Let S be closed and hyperbolic. Let $\phi \in \text{Homeo}^+(S)$ and suppose that, for every simple closed curve α , $\phi \circ \alpha$ is homotopic to α . Show that ϕ is isotopic to the identity.
8. Let α, β be a pair of simple closed curves on a surface S , such that $i(\alpha, \beta) = 1$.
 - (a) Prove that $T_\alpha T_\beta(\alpha) = \beta$.
 - (b) Prove the *braid relation*: $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta$.
9. Let $S = S_{1,0,1}$, the torus with one boundary component. Let α, β be a standard pair of simple closed curves on S such that $i(\alpha, \beta) = 1$, and let γ be the boundary curve. Prove that $T_\gamma = (T_\alpha T_\beta)^6$.
10. Let S be the closed surface of genus $g \geq 3$. Prove that the centre of $\text{Mod}(S)$ is trivial. [*Hint: consider a suitable collection of simple closed curves on S . For instance, when $g = 3$, consider the curves in the following picture.*]



11. The group $SL_2(\mathbb{Z})$ has a presentation

$$\langle a, b \mid aba = bab, (ab)^6 = 1 \rangle$$

where a and b correspond to the elementary matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

respectively. Prove that

$$\text{Mod}(S_{1,0,1}) \cong \langle a, b \mid aba = bab \rangle.$$