

## Introduction to non linear Analysis

## Example sheet n° 3 - NLS

Exercices to be done : 1-2-3.

**Exercise 1** (A symmetry of the harmonic oscillator). *We consider the cubic non linear harmonic oscillator in dimension 2*

$$i\partial_t u + \Delta u - |x|^2 u + u|u|^2 = 0.$$

1. Define the renormalization

$$\begin{cases} u(t, x) = \frac{1}{L} w(s, y) \\ w(s, y) = e^{-i \frac{b(s)|y|^2}{4}} v(s, y) \\ y = \frac{x}{L}, \quad \frac{ds}{dt} = \frac{1}{L^2}. \end{cases}$$

Show that  $v(s, y)$  satisfies the same equation iff

$$\begin{cases} \frac{L_s}{L} + b = 0 \\ \frac{b_s}{4} - \left( \frac{b^2}{4} + L^4 \right) - \frac{b}{2} \frac{L_s}{L} = -1 \end{cases} \quad (0.1)$$

where  $f_s = \frac{df}{ds}$ .

2. Find a conserved quantity and draw the phase portrait of (0.1).

**Exercise 2** (Local existence in  $H^2$  for cubic (NLS) in dimension 2). *We consider*

$$(S_\varepsilon) \quad i\partial_t u + \Delta u + \varepsilon |u|^2 u = 0 \quad \text{dans } \mathbb{R} \times \mathbb{R}^2, \quad \varepsilon \in \{-1, 1\}$$

with data  $u_0 \in H^2(\mathbb{R}^2)$ .

Let  $E_T = \mathcal{C}([0, T]; H^2)$  and  $u_L = S(t)u_0$ . For  $u \in E_T$ , we let

$$\forall t \in [0, T], \quad \Phi(u)(t) = u_L(t) + i\varepsilon \int_0^t S(t-\tau)(|u|^2 u(\tau)) d\tau.$$

1. Show that  $H^2(\mathbb{R}^2)$  is stable by product.

2. Show that  $\Phi$  is well defined from  $E_T$  into  $E_T$ , and there exist two constants  $C_1, C_2 > 0$  such that for all  $u, v$  in  $B_{E_T}(u_L, R)$ ,

$$\|\Phi(u) - u_L\|_{E_T} \leq C_1 T(R^3 + \|u_0\|_{H^2}^3) \quad \text{and} \quad \|\Phi(u) - \Phi(v)\|_{E_T} \leq C_2 T(R^2 + \|u_0\|_{H^2}^2) \|u - v\|_{E_T}.$$

3. Conclude that there exists  $c > 0$  and a time  $T \geq c/\|u_0\|_{H^2}^2$  such that  $\Phi$  has a fixed point in  $E_T$ .

4. Conclude that there exists  $T^* > 0$  such that  $(S_\varepsilon)$  with data  $u_0 \in H^2$  has a unique maximal solution  $u \in \mathcal{C}([0, T^*]; H^2) \cap \mathcal{C}^1([0, T^*]; L^2)$ .

5. In this question, we look for a blow up criterion.

(a) Prove the Gagliardo-Nirenberg estimate :

$$\forall u \in H^2(\mathbb{R}^2), \quad \|\partial_1 u\|_{L^4}^2 \leq 3 \|u\|_{L^\infty} \|\partial_{11}^2 u\|_{L^2}.$$

(b) Prove the tame estimate :

$$\forall (u, v) \in H^2 \times H^2, \quad \|uv\|_{H^2} \leq C_0 (\|u\|_{L^\infty} \|v\|_{H^2} + \|v\|_{L^\infty} \|u\|_{H^2})$$

for some universal constant  $C_0$ .

(c) Show that there exists  $C > 0$  universal such that for all solution  $u \in E_T$  of  $(S_\varepsilon)$  :

$$\forall t \in [0, T], \|u(t)\|_{H^2} \leq \|u_0\|_{H^2} + C \int_0^t \|u\|_{L^\infty}^2 \|u\|_{H^2} d\tau.$$

(d) Conclude that  $T^* < +\infty$  implies  $\int_0^{T^*} \|u(t)\|_{L^\infty}^2 dt = +\infty$ . For  $\varepsilon = -1$ , does this allow to conclude  $T^* = +\infty$  ?

6. Prove the Brezis-Gallouët inequality :

$$\|u\|_{L^\infty} \leq C \|u\|_{H^1} \left[ 1 + \sqrt{\log \left( 1 + \frac{\|u\|_{H^2}}{\|u\|_{H^1}} \right)} \right]$$

for all  $u \in H^2(\mathbb{R}^2)$  not identically zero. (Hint : estimate  $\|\hat{u}\|_{L^1}$  by splitting in low and high frequencies and optimize the localization parameter).

7. Using Theorem 6.2.1 in the notes (gwp in  $H^1$ ), prove global existence in  $H^2$  for  $\varepsilon = -1$ . (Hint : use a logarithmically improved Gronwall lemma)

**Exercise 3** (An upper bound on blow up rate for (NLS)). We work in  $\mathbb{R}^2$ . Let the focusing (NLS)

$$\begin{cases} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0, x) = u_0(x) \end{cases}, \quad x \in \mathbb{R}^2, \quad 3 < p < 5.$$

Let  $H_r^1$  be the set of  $H^1$  functions with radial symmetry, then the Cauchy problem is well posed in  $H_r^1$ . We pick  $u_0 \in H_r^1$  and assume that the solution blows up in finite time  $0 < T < +\infty$ . The aim of this problem is to derive an upper bound on  $\|\nabla u(t)\|_{L^2}$  as  $t \uparrow T$ .

Integration by parts should be done without boundary terms (without justification). We let

$$s_c = 1 - \frac{2}{p-1}$$

and  $E_0$  be the energy of the data. We recall Young's inequality :

$$|xy| \leq \frac{1}{p} \left( \frac{|x|}{A} \right)^p + \frac{(A|y|)^{p'}}{p'}, \quad 1 \leq p, p' \leq +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad A > 0. \quad (0.2)$$

1. Let  $\chi \in C_c^\infty(\mathbb{R}^2)$  with spherical symmetry, prove the formulas :

$$\frac{1}{2} \frac{d}{d\tau} \int \chi |u|^2 = \text{Im} \left( \int \nabla \chi \cdot \nabla u \bar{u} \right),$$

and

$$\frac{1}{2} \frac{d}{d\tau} \text{Im} \left( \int \nabla \chi \cdot \nabla u \bar{u} \right) = \int \chi'' |\nabla u|^2 - \frac{1}{4} \int \Delta^2 \chi |u|^2 - \left( \frac{1}{2} - \frac{1}{p+1} \right) \int \Delta \chi |u|^{p+1}.$$

2. Prove that for all  $u \in H_r^1$ ,

$$\forall R > 0, \quad \|u\|_{L^\infty(r \geq R)}^2 \leq \frac{2}{R} \|u\|_{L^2} \|\nabla u\|_{L^2}.$$

3. Let  $R > 0$ ,  $\psi \in C_c^\infty(\mathbb{R}^2)$  with spherical symmetry and

$$\psi(x) = \begin{cases} \frac{|x|^2}{2} & \text{pour } |x| \leq 2 \\ 0 & \text{pour } |x| \geq 10 \\ \psi'' \leq 1. \end{cases}$$

Let

$$\chi(x) = \psi_R(x) = R^2 \psi \left( \frac{x}{R} \right),$$

show that

$$c(d, p) \int |\nabla u|^2 + \frac{1}{2} \frac{d}{dt} \Im \left( \int \nabla \psi_R \cdot \nabla u \bar{u} \right) \leq C(d, p) \left[ |E_0| + \int_{|x| \geq 2R} |u|^{p+1} + \frac{1}{R^2} \int_{2R \leq |x| \leq 10R} |u|^2 \right]$$

for some constants  $c(d, p), C(d, p) > 0$  independent of  $R$ .

4. Prove using (0.2) that :

$$\frac{c(d,p)}{2} \int |\nabla u|^2 + \frac{1}{2} \frac{d}{dt} \Im \left( \int \nabla \psi_R \cdot \nabla u \bar{u} \right) \leq C(u_0, d, p) \left[ 1 + \frac{1}{R^2} + \frac{1}{R^{\frac{2}{\alpha}}} \right] \quad (0.3)$$

with

$$\alpha = \frac{5-p}{p-1}.$$

5. Integrate in time (0.3) and prove :  $\forall 0 < t_0 < t_2 < T$ ,

$$\int_{t_0}^{t_2} (t_2 - t) \|\nabla u(t)\|_{L^2}^2 dt \leq C(u_0, d, p) \left[ \frac{(t_2 - t_0)^2}{R^{\frac{2}{\alpha}}} + R(t_2 - t_0) \|\nabla u(t_0)\|_{L^2} + R^2 \right].$$

6. Choose  $R = (T - t_0)^{\frac{\alpha}{1+\alpha}}$  and conclude that for  $t$  close enough to  $T$  :

$$\int_{t_0}^T (T - t) \|\nabla u(t)\|_{L^2}^2 dt \leq C(d, p, u_0) (T - t_0)^{\frac{2\alpha}{1+\alpha}} + (T - t_0)^2 \|\nabla u(t_0)\|_{L^2}^2.$$

7. Show that for  $t_0$  close enough to  $T$  :

$$\int_{t_0}^T (T - t) \|\nabla u(t)\|_{L^2}^2 dt \leq C(d, p, u_0) (T - t_0)^{\frac{2\alpha}{1+\alpha}}.$$

8. Conclude that there exists a sequence  $t_n \rightarrow T$  such that

$$\|\nabla u(t_n)\|_{L^2} \leq \frac{C(d, p, u_0)}{(T - t_n)^{\frac{1}{1+\alpha}}}.$$

(remark : this bound is sharp !).

9. Open problem : prove any bound on blow up rate in the critical case  $p = 3$  !