Introduction to non linear Analysis

Example sheet n° 4 - Variational methods

Exercices to be done: 1-2.

Exercice 1 (Ground state of a gaseous star)**.** *We work in* $\mathbb{R}^3$. *To every positive function* $u : \mathbb{R}^3 \to \mathbb{R}^+$, *we associate its Poisson field*

$$E_u = \nabla \phi_u \quad with \quad \phi_u = -\frac{1}{4\pi|x|} \star u.$$

The potential ϕ_u *is a solution to*

$$\Delta \phi_u = u. \tag{0.1}$$

We admit the Hardy-Littlewood-Sobolev inequality which is the borderline case of Young's inequality in $\mathbb{R}^d$ *(see the notes for a proof): let* $0 < \alpha < d$, $1 < p, r < +\infty$ *with*

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$$

then

$$\|\frac{1}{|\cdot|^\alpha} \star f\|_{L^r(\mathbb{R}^d)} \leq C_{r,p}\|f\|_{L^p(\mathbb{R}^d)}.$$

1. *Show that*

$$|E_u(x)| \lesssim \frac{1}{|x|^2} \star |u|$$

   and conclude

$$\|E_u\|_{L^2} \lesssim \|u\|_{L^2}^{\frac{1}{3}} \|u\|_{L^1}^{\frac{2}{3}}.$$

2. *Compute* $\widehat{E_u}$ *in terms of* $\widehat{u}$ *and conclude*

$$\|E_u\|_{H^1} \lesssim \|u\|_{L^2} + \|u\|_{L^1}.$$

3. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^1 \cap L^2$ *such that*

$$u_n \rightharpoonup u \quad dans \quad L^2.$$

   Show using Plancherel that

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^3), \quad \int E_{u_n}\overline{\phi}\, dx \longrightarrow \int E_u \overline{\phi}\, dx.$$

   Prove that

$$E_{u_n} \rightharpoonup E_u \quad dans \quad L^2.$$

4. *We assume that u has spherical symmetry. Show the representation formula*

$$E_u(r) = \phi_u'(r)e_r = \left(\frac{1}{r^2}\int_0^r \tau^2 u(\tau)d\tau\right)\frac{x}{|x|}.$$

   Show that

$$\forall R > 0, \quad \int_{|x|\geq R} |E_u|^2 dx \lesssim \frac{\|u\|_{L^1}^2}{R}.$$

5. *Let* $(u_n)_{n\in\mathbb{N}}$ *be a bounded sequence in* $L^1 \cap L^2$ *of radially symmetric positive functions. Show that we can extract* $(u_{\varphi(n)})_{n\in\mathbb{N}}$ *such that*

$$u_{\varphi(n)} \rightharpoonup u \quad in \quad L^2$$

   and

$$E_{u_{\varphi(n)}} \to E_u \quad in \quad L^2.$$

6. *Let $M > 0$ and*
$$A(M) = \left\{ u : \mathbb{R}^3 \mapsto \mathbb{R}^+ \quad \text{with} \quad u \in L^2(\mathbb{R}^3) \ \text{and} \ \int_{\mathbb{R}^3} u \, dx = M \right\}.$$
Let
$$I(M) = \inf_{u \in A(M)} \left[\int_{\mathbb{R}^3} |u|^2 \, dx - \int_{\mathbb{R}^3} |E_u|^2 \, dx \right].$$
Show that
$$-\infty < I(M) < 0.$$

7. *Compute $I(M)$ in terms of M and $I(1)$.*

8. *Let $A_{rad}(M)$ be the set of radially symmetric elements $u \in A(M)$. Let*
$$I_{rad}(M) = \inf_{u \in A_{rad}(M)} \left[\int_{\mathbb{R}^3} |u|^2 \, dx - \int_{\mathbb{R}^3} |E_u|^2 \, dx \right].$$
Show that $I_{rad}(M)$ is attained.

Exercice 2 (Orbital stability of the ground state in the mass critical case)**.** *Let $d = 2$, $p = 3$ and consider the focusing (NLS). Let Q be the ground state.*

1. *Let u_n be a sequence in H^1 with*
$$\left| \begin{array}{l} \forall n \geq 1, \quad \|\nabla u_n\|_{L^2} = \|\nabla Q\|_{L^2} \\ \lim_{n \to +\infty} \|u_n\|_{L^2} = \|Q\|_{L^2}. \end{array} \right.$$
show that there exists $x_n \in \mathbb{R}^d$, $\gamma \in \mathbb{R}$ such that up to a subsequence
$$u_n(\cdot + x_n) \to Q e^{i\gamma} \quad \text{in} \quad H^1.$$
Hint: use the profile decomposition.

2. *Show the following "orbital stability" statement: $\forall \varepsilon > 0$, $\exists \eta > 0$ such that forall $u_0 \in H^1$ with $|\|u_0\|_{L^2} - \|Q\|_{L^2}| < \eta$, let $u \in \mathcal{C}([0,T), H^1)$ be the corresponding unique solution to (NLS), then if $T < +\infty$, there exist $0 < T^* < T$ such that $\forall t \in [T^*, T)$, $\exists x(t) \in \mathbb{R}^d$, $(\lambda(t), x(t), \gamma(t)) \in \times \mathbb{R}_*^+ \times \mathbb{R}^d \times \mathbb{R}$ and $v(t, \cdot) \in H^1$ such that*
$$u(t,x) = \frac{1}{\lambda(t)} \left[Q + v \right] \left(t, \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \quad \text{with} \quad \|v(t, \cdot)\|_{H^1} < \varepsilon.$$
Hint: Argue by contradiction affter renomalization of the sequence $u_n(t_n, x)$.

Exercice 3 (Kinetic model of stars)**.** *A galaxy is a cluster of typically 10^{15} stars. A statistic description is given by the distribution $f(x,v) > 0$ which is the density of stars which have the speed $v \in \mathbb{R}^3$ at the point $x \in \mathbb{R}^3$. The total number of stars at $x \in \mathbb{R}^3$ is therefore*
$$\rho_f(x) = \int_{v \in \mathbb{R}^3} f(x,v) \, dv,$$
and the total number of stars is
$$\|f\|_{L^1(\mathbb{R}^6)} = \int_{\mathbb{R}^6} f(x,v) \, dx \, dv = \int_{\mathbb{R}^3} \rho_f(x) \, dx.$$
The total kinetic energy of the galaxy is
$$E_{cin}(f) = \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f(x,v) \, dx \, dv.$$
Last, stars are submitted only to the gravitational force, and the total potential energy is
$$E_{pot}(f) = \int_{\mathbb{R}^3} |\nabla \phi_f(x)|^2 \, dx \quad \text{où} \quad \phi_f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_f(y)}{|x-y|} \, dy.$$

Given $M_1, M_2 > 0$, we consider the minimization problem:

$$I(M_1, M_2) = \inf_{f \in \mathcal{A}(M_1.M_2)} E(f)$$

which defines a stable galaxy, where

$$A(M_1.M_2) = \left\{ f(x,v) \geq 0, \quad \|f\|_{L^1(\mathbb{R}^6)} = M_1, \quad \|f\|_{L^2(\mathbb{R}^6)} = M_2 \right\}$$

and

$$E(f) = \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv - \int_{\mathbb{R}^3} |\nabla \phi_f(x)|^2 \, dx.$$

1. *Let $x \in \mathbb{R}^3$. By splitting $|v| \leq R$ et $|v| \geq R$, show that*

$$|\rho_f(x)| \lesssim R^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} f^2(x,v) \, dv \right)^{\frac{1}{2}} + \frac{1}{R^2} \int_{\mathbb{R}^3} |v|^2 f(x,v) \, dv.$$

2. *Conclude by optimizing on R that*

$$\forall x \in \mathbb{R}^3, \quad |\rho_f(x)| \lesssim \left(\int_{\mathbb{R}^3} |v|^2 f(x,v) \, dv \right)^{\frac{3}{7}} \left(\int_{\mathbb{R}^3} f^2(x,v) \, dv \right)^{\frac{2}{7}}.$$

3. *Prove using Hölder:*

$$\|\rho_f\|_{L^{\frac{7}{5}}(\mathbb{R}^3)} \lesssim \||v|^2 f\|^{\frac{3}{7}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{4}{7}}_{L^2(\mathbb{R}^6)}.$$

4. *Prove using Hölder:*

$$\|\rho_f\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)} \lesssim \||v|^2 f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{5}{6}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^6)}.$$

5. *Show that*

$$|\nabla \phi_f(x)| \lesssim \frac{1}{|x|^2} \star \rho_f$$

and obtain the interpolation estimate

$$\int |\nabla \phi_f(x)|^2 \, dx \lesssim \||v|^2 f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{5}{6}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^6)}.$$

6. *Show that*

$$I(M_1, M_2) > -\infty.$$

7. *Using the scaling*

$$f(x,v) = f\left(\frac{x}{\lambda}, \lambda v\right), \quad \lambda > 0$$

show that

$$I(M_1, M_2) < 0.$$

8. *Using the scaling*

$$f_{,\mu}(x,v) = \frac{\mu}{2} f\left(\frac{x}{\lambda}, \mu v\right), \quad \lambda, \mu > 0,$$

show that $I(M_1, M_2)$ is homogeneous in both M_1 and M_2.

The compactness of the minimizing problem can be proved, but this requires more work...