Introduction to non linear Analysis

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Example sheet n° 3 - NLS

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Exercices to be done : 1-2-3.

Exercice 1 (A symmetry of the harmonic oscillator)**.** *We consider the cubic non linear harmonic oscillator in dimension 2*

$$i\partial_t u + \Delta u - |x|^2 u + u|u|^2 = 0.$$

1. *Define the renormalization*

$$\left|\begin{array}{l} u(t,x) = \frac{1}{L} w(s,y) \\ w(s,y) = e^{-i\frac{b(s)|y|^2}{4}} v(s,y) \\ y = \frac{x}{L}, \quad \frac{ds}{dt} = \frac{1}{L^2}. \end{array}\right.$$

Show that $v(s,y)$ satisfies the same equation iff

$$\left|\begin{array}{l} \frac{L_s}{L} + b = 0 \\ \frac{b_s}{4} - \left(\frac{b^2}{4} + L^4\right) - \frac{b}{2}\frac{L_s}{L} = -1 \end{array}\right. \tag{0.1}$$

where $f_s = \frac{df}{ds}$.

2. *Integrate the dynamical system* (0.1) *in time t. (hint : look for a conserved quantity and draw the phase portrait).*

Exercice 2 (Local existence in H^2 for cubic (NLS) in dimension 2)**.** *We consider*

$$(S_\varepsilon) \qquad\qquad i\partial_t u + \Delta u + \varepsilon|u|^2 u = 0 \quad dans \ \ \mathbb{R} \times \mathbb{R}^2, \qquad \varepsilon \in \{-1,1\}$$

with data $u_0 \in H^2(\mathbb{R}^2)$.

Let $E_T = \mathcal{C}([0,T];H^2)$ and $u_L = S(t)u_0$. For $u \in E_T$, we let

$$\forall t \in [0,T], \ \Phi(u)(t) = u_L(t) + i\varepsilon \int_0^t S(t-\tau)((|u|^2 u(\tau))\,d\tau.$$

1. *Show that $H^2(\mathbb{R}^2)$ is stable by product.*

2. *Show that Φ is well defined from E_T into E_T, and there exist two constants $C_1, C_2 > 0$ such that for all u, v in $B_{E_T}(u_L, R)$,*

$$\|\Phi(u) - u_L\|_{E_T} \le C_1 T(R^3 + \|u_0\|_{H^2}^3) \ \ and \ \ \|\Phi(u) - \Phi(v)\|_{E_T} \le C_2 T(R^2 + \|u_0\|_{H^2}^2)\|u - v\|_{E_T}.$$

3. *Conclude that there exists $c > 0$ and a time $T \ge c/\|u_0\|_{H^2}^2$ such that Φ has a fixed point in E_T.*

4. *Conclude that there exists $T^* > 0$ such that (S_ε) with data $u_0 \in H^2$ has a unique maximal solution $u \in \mathcal{C}([0,T^*[;H^2) \cap \mathcal{C}^1([0,T^*[;L^2)$.*

5. *In this question, we look for a blow up criterion.*

   (a) *Prove the Gagliardo-Nirenberg estimate :*

   $$\forall u \in H^2(\mathbb{R}^2), \ \|\partial_1 u\|_{L^4}^2 \le 3\|u\|_{L^\infty}\|\partial_{11}^2 u\|_{L^2}.$$

   (b) *Prove the* tame *estimate :*

   $$\forall (u,v) \in H^2 \times H^2, \ \|uv\|_{H^2} \le C_0\big(\|u\|_{L^\infty}\|v\|_{H^2} + \|v\|_{L^\infty}\|u\|_{H^2}\big)$$

   for some universal constant C_0.

(c) Show that there exists $C > 0$ universal such that for all solution $u \in E_T$ of (S_ε) :

$$\forall t \in [0, T], \ \|u(t)\|_{H^2} \leq \|u_0\|_{H^2} + C \int_0^t \|u\|_{L^\infty}^2 \|u\|_{H^2} \, d\tau.$$

(d) Conclude that $T^* < +\infty$ implies $\int_0^{T^*} \|u(t)\|_{L^\infty}^2 \, dt = +\infty$. For $\varepsilon = -1$, does this allow to conclude $T^* = +\infty$?

6. Using Theorem 6.2.1 in the notes (gwp in H^1), prove global existence in H^2 for $\varepsilon = -1$.

Exercice 3 (An upper bound on blow up rate for (NLS)). *We work in $\mathbb{R}^2$. Let the focusing (NLS)*

$$\left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0, x) = u_0(x) \end{array} \right. \quad , \quad x \in \mathbb{R}^2, \ \ 3 < p < 5.$$

Let H_r^1 be the set of H^1 functions with radial symmetry, then the Cauchy problem is well posed in H_r^1. We pick $u_0 \in H_r^1$ and assume that the solution blows up in finite time $0 < T < +\infty$. The aim of this problem is to derive an upper bound on $\|\nabla u(t)\|_{L^2}$ as $t \uparrow T$.

Integration by parts should be done without boundary terms (without justification). We let

$$s_c = 1 - \frac{2}{p-1}$$

and E_0 be the energy of the data. We recall Young's inequality :

$$|xy| \leq \frac{1}{p} \left(\frac{|x|}{A} \right)^p + \frac{(A|y|)^{p'}}{p'}, \quad 1 \leq p, p' \leq +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad A > 0. \tag{0.2}$$

1. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with spherical symmetry, prove the formulas :

$$\frac{1}{2} \frac{d}{d\tau} \int \chi |u|^2 = Im \left(\int \nabla \chi \cdot \nabla u \bar{u} \right),$$

and

$$\frac{1}{2} \frac{d}{d\tau} Im \left(\int \nabla \chi \cdot \nabla u \bar{u} \right) = \int \chi'' |\nabla u|^2 - \frac{1}{4} \int \Delta^2 \chi |u|^2 - \left(\frac{1}{2} - \frac{1}{p+1} \right) \int \Delta \chi |u|^{p+1}.$$

2. Prove that for all $u \in H_r^1$,

$$\forall R > 0, \quad \|u\|_{L^\infty(r \geq R)}^2 \leq \frac{2}{R} \|u\|_{L^2} \|\nabla u\|_{L^2}.$$

3. Let $R > 0$, $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with spherical symmetry and

$$\psi(x) = \left| \begin{array}{ll} \frac{|x|^2}{2} & pour \ |x| \leq 2 \\ 0 & pour \ |x| \geq 3 \end{array} \right. .$$

Let

$$\chi(x) = \psi_R(x) = R^2 \psi \left(\frac{x}{R} \right),$$

show that

$$c(d, p) \int |\nabla u|^2 + \frac{1}{2} \frac{d}{dt} \Im \left(\int \nabla \psi_R \cdot \nabla u \bar{u} \right) \leq C(d, p) \left[|E_0| + \int_{|x| \geq R} |u|^{p+1} + \frac{1}{R^2} \int_{2R \leq |x| \leq 3R} |u|^2 \right]$$

for some constants $c(d, p), C(d, p) > 0$ independant of R.

4. Prove using (0.2) that :

$$\frac{c(d, p)}{2} \int |\nabla u|^2 + \frac{1}{2} \frac{d}{dt} \Im \left(\int \nabla \psi_R \cdot \nabla u \bar{u} \right) \leq C(u_0, d, p) \left[1 + \frac{1}{R^2} + \frac{1}{R^{\frac{2}{\alpha}}} \right] \tag{0.3}$$

with

$$\alpha = \frac{5-p}{p-1}.$$

5. *Integrate in time (0.3) and prove :* $\forall 0 < t_0 < t_2 < T,$

$$\int_{t_0}^{t_2} (t_2 - t)\|\nabla u(t)\|_{L^2}^2 dt \leq C(u_0, d, p) \left[\frac{(t_2 - t_0)^2}{R^{\frac{2}{\alpha}}} + R(t_2 - t_0)\|\nabla u(t_0)\|_{L^2} + R^2 \right].$$

6. *Choose* $R = (T - t_0)^{\frac{\alpha}{1+\alpha}}$ *and conclude that for t close enough to* T :

$$\int_{t_0}^{T} (T - t)\|\nabla u(t)\|_{L^2}^2 dt \leq C(d, p, u_0)(T - t)^{\frac{\alpha}{2+\alpha}} + (T - t_0)^2\|\nabla u(t_0)\|_{L^2}^2.$$

7. *Show that for t close enough to* T :

$$\int_{t_0}^{T} (T - t)\|\nabla u(t)\|_{L^2}^2 dt \leq C(d, p, u_0)(T - t)^{\frac{2\alpha}{1+\alpha}}.$$

8. *Conclude that there exists a sequence* $t_n \to T$ *such that*

$$\|\nabla u(t_n)\|_{L^2} \leq \frac{C(d, p, u_0)}{(T - t_n)^{\frac{1}{1+\alpha}}}.$$

(remark : this bound is sharp !).

9. *Open problem : prove any bound on blow up rate in the critical case* $p = 3$ *!*