# An introduction to the study of non linear waves

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October 9, 2020

# Chapter 1

# Lebesgue spaces

This chapter is devoted to the derivation of fundamental properties of Lebesgue spaces $L^p(\mathbb{R}^d)$. After recalling classical inequalities (Hölder, Minkowski and Young), we introduce the complex interpolation method which is a powerful tool to derive estimates. We then give a self contained proof of the critical Hardy-Littlewood-Sobolev estimates which are essential for many applications in mathematical physics and are a first intrusion into harmonic analysis methods.

## 1.1 Banach space structure and duality

In this section, we briefly recall basis classical properties of Lebesgue spaces in a general measured topological space $(X, \mathcal{O}, \mu)$. Classical references on Lebesgue spaces are [3], [17] et [34].

### 1.1.1 Banach space structure

Let us recall the definition of Lebesgue spaces.

**Definition 1.1.1.** *Let $(X,\mu)$ be a measured topological space. If $1 \leq p < +\infty$ then $L^p = L^p(X,\mu)$ is the space of equivalence classes (for the $\mu$ almost everywhere equality) of borelian functions $f$ on $X$ with values in $\mathbb{R}$ ou $\mathbb{C}$ such that $|f|^p$ is integrable. We let*

$$\|f\|_{L^p} \stackrel{def}{=} \left(\int_X |f(x)|^p d\mu(x)\right)^{\frac{1}{p}}.$$

*If $p = +\infty$, we define $L^\infty(X,\mu)$ as the set of equivalence classes for borelian functions $f$ on $X$ such that the set of $\lambda > 0$ satisfying $\mu\big(\{x \in X \,/\, |f(x)| > \lambda\}\big) > 0$ is bounded. We let*

$$\|f\|_{L^\infty} \stackrel{def}{=} \sup\Big\{\lambda > 0 \,/\, \mu\big(\{x \in X \,/\, |f(x)| > \lambda\}\big) > 0\Big\}\cdot$$

The following Theorem is fundamental and relies on the construction of the measure.

**Theoreme 1.1.1** (Banach space structure). *$(L^p(X,\mu), \|\cdot\|_{L^p})$ is a Banach space.*

The vectorial space structure follows for $1 \leq p < \infty$ from

$$|f(x)+g(x)|^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p).$$

The rest of the proof (see e.g. [5, 34] for détails) relies essentially on Hölder's inequality.

**Lemma 1.1.1** (Hölder's inequality)**.** *Let* $p \in [1, +\infty]$*. Let* $p' \stackrel{def}{=} \dfrac{p}{p-1}$ *be the conjuguate exponent (with the rule* $1/0 = \infty$*). Then*

$$\forall f, g \in L^p \times L^{p'}, \quad \|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}.$$

*Proof of Lemma 1.1.1.* It is obvious if $p$ is 1 or $\infty$. In the other cases, the concavity of the log function ensures that for $(a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\theta \in [0,1]$,

$$\theta \log a + (1-\theta) \log b \leq \log(\theta a + (1-\theta) b).$$

Exponentiating yields

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b. \tag{1.1}$$

By homogeneity, we may without loss of generality assume $\|f\|_{L^p} = \|g\|_{L^{p'}} = 1$. Applying the above inequality with $\theta = 1/p$, $a = |f(x)|^p$ et $b = |g(x)|^{p'}$ yields

$$|f(x)|\,|g(x)| \quad = \quad (|f(x)|^p)^{\frac{1}{p}} (|g(x)|^{p'})^{\frac{1}{p'}} \leq \frac{1}{p} |f(x)|^p + \Big(1 - \frac{1}{p}\Big) |g(x)|^{p'}.$$

which integration with respect to $\mu$ concludes the proof. □

*Remark.* Hölder implies that $\|\cdot\|_{L^p}$ is a norm. Indeed,

$$\begin{aligned} |f(x)+g(x)|^p &= |f(x)+g(x)|\,|f(x)+g(x)|^{p-1} \\ &\leq |f(x)|\,|f(x)+g(x)|^{p-1} + |g(x)|\,|f(x)+g(x)|^{p-1}. \end{aligned}$$

Since $f+g$ belongs to $L^p$, $|f+g|^{p-1}$ belongs to $L^{p'}$, and from Hölder:

$$\int_\Omega |f(x)+g(x)|^p d\mu(x) \leq (\|f\|_{L^p} + \|g\|_{L^p}) \Big( \int_\Omega |f(x)+g(x)|^p d\mu(x) \Big)^{1-\frac{1}{p}}.$$

We conclude that $\|\cdot\|_{L^p}$ satisfies the triangle inequality (known as *Minkowski's inequality*) :

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}.$$

The proof that $(L^p, \|\cdot\|_{L^p})$ is complete is then very similar to the ones of Theorems 3.2.4 p. 81 and 3.3.2 p 87 of [3], and is detailed in [34].

The following variant of Hölder are very useful, the proof is left to the reader.

**Corollary 1.1.1** (Hölder type inequalities)**.** *There holds:*

*(i) let* $1 \leq p, q \leq \infty$*,* $0 \leq \theta \leq 1$*, then*

$$\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta} \quad \textit{with} \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q} \cdot;$$

*(ii) let* $1 \leq p_1, \cdots, p_N, r \leq +\infty$ *then*

$$\|\prod_{i=1}^N f_i\|_{L^r} \leq \prod_{i=1}^N \|f_i\|_{L^{p_i}} \quad \textit{with} \quad \frac{1}{r} = \sum_{i=1}^N \frac{1}{p_i} \cdot$$

The separability of $L^p$ for $p < +\infty$ relies on classical density arguments, see [34].

**Proposition 1.1.1** (Separability). *For $1 \leq p < +\infty$, for every borelian set $A$ of $\mathbb{R}^d$ and measure $\mu$ absolutely continuous with respect to the Lebesgue measure, the space $L^p(A,\mu)$ is separable. Moreover, simple functions* [1] *are dense in $L^p(A,\mu)$, and so is the set of continuous functions with compact support in the closure of $A$.*

Let us insist that the result if false for $p = +\infty$, and the limit cases $p \in \{1, +\infty\}$ should always be treated with caution.

## 1.2 Complex interpolation

We present in this section a technical tool, elementary but very powerful, known as complex interpolation. We will give application of the method when proving Strichartz estimates for the free Schrödinger equation. Other related methods of real interpolation can be found in [33].

### 1.2.1 Riesz-Thorin interpolation Theorem

The main result of complex interpolation is the following:

**Theoreme 1.2.1** (Riesz-Thorin). *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. Let $(X,\mu)$ et $(Y,\nu)$ be two measured space. Let $T$ be a linear operator from $L^{p_0}(X,\mu) + L^{p_1}(X,\mu)$ into $L^{q_0}(Y,\nu) + L^{q_1}(Y,\nu)$, which is bounded from $L^{p_0}(X,\mu)$ into $L^{q_0}(Y,\nu)$ and from $L^{p_1}(X,\mu)$ into $L^{q_1}(Y,\nu)$. Assume that there exists $\theta \in ]0,1[$ such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad et \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

*then $T$ is also bounded from $L^p(X,\mu)$ into $L^q(Y,\nu)$ with*

$$\|T\|_{\mathcal{L}(L^p;L^q)} \leq \|T\|^{1-\theta}_{\mathcal{L}(L^{p_0};L^{q_0})} \|T\|^{\theta}_{\mathcal{L}(L^{p_1};L^{q_1})}.$$

The proof relies on the maximum principle for holomorphic functions.

**Lemma 1.2.1** (Phragmen-Lindelöf). *Let $F$ be a function of the complex variable which is continuous and bounded in the band*

$$S \overset{def}{=} \big\{x + iy \,/\, x \in [0,1],\, y \in \mathbb{R}\big\}$$

*and holomorphic in the interior of $S$. Let*

$$M_0 = \sup_{y\in\mathbb{R}} |F(iy)| \quad and \quad M_1 = \sup_{y\in\mathbb{R}} |F(1+iy)|.$$

*Then*

$$\forall (x,y) \in [0,1] \times \mathbb{R}, \quad |F(x+iy)| \leq M_0^{1-x} M_1^x.$$

*Proof of Lemma 1.2.1.* By possibly perturbing $F$ by a constant which can be chosen arbitrarily small, we may assume that $M_0, M_1$ are non negative. Let $G(z) = M_0^{z-1} M_1^{-z} F(z)$, then $G$ has the same properties of $F$ and is bounded by 1 on the boundary of $S$. We need to prove that $G$ is bounded by 1 on all $S$. Let the sequence $(G_n)_{n\geq 1}$ defined on $S$ by

$$G_n(z) = G(z) \exp\left(\frac{z^2-1}{n}\right).$$

[1] ie finite linear combination of characteristic function of mesureable disjoint sets of finite measure

Since $G$ is bounded on $S$, $(G_n)_{n\in\mathbb{N}}$ converges pointwise to $G$ on $S$, hence we need only prove that $G_n$ is bounded by 1 for $n$ large enough. For all $n \in \mathbb{N}$, $G_n$ is continuous bounded on the whole rectangle $\{x+iy \,/\, 0 \le x \le 1,\ |y| \le N\}$, holomorphic inside this rectangle. The maximum principle ensures that $G_n$ attains its maximum on the boundary of the rectangle. Since $G$ is bounded on $S$ and $|G_n(z)| \le |G(z)|\exp(-y^2/n)$, we conclude that if $N$ has been chosen large enough then $|G_n|$ does not exceed 1 on the boundary of the rectangle, and hence is bounded by 1 on the whole rectangle. Letting $N \to +\infty$, we conclude that $G_n$ is bounded on the whole band $S$. $\square$

*Proof of Theorem 1.2.1.* By duality (Lemma 2.2.4), we equivalently need to prove

$$\forall f \in L^p(X,\mu),\ \forall g \in L^{q'}(Y,\nu),\ \left|\int_Y T(f)\, g\, d\nu\right| \le M_0^\theta M_1^{1-\theta} \|f\|_{L^p} \|g\|_{L^{q'}}. \tag{1.2}$$

From Proposition 1.1.1, we need only prove the result when $f$ and $g$ are simple functions (at least when $p$ and $q'$ are finite), and we may therefore assume that

$$f = \sum_{j=1}^{p} a_j 1_{A_j} \quad \text{and} \quad g = \sum_{k=1}^{q} b_k 1_{B_k}$$

where the coefficients $a_j$ et $b_k$ are all non zero and the sets $A_j$ and $B_k$ are $\mu$-mesurables. Up to renormalization, we may assume $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Let $z \in S$ as in the Phragmen-Lindelöf Lemma, let

$$f_z = \sum_{j=1}^{p} \frac{a_j}{|a_j|} |a_j|^{p(\frac{1-z}{p_0} + \frac{z}{p_1})} 1_{A_j} \quad \text{and} \quad g_z = \sum_{k=1}^{q} \frac{b_k}{|b_k|} |b_k|^{q'(\frac{1-z}{q_0'} + \frac{z}{q_1'})} 1_{B_k}.$$

then $f_\theta = f$, $g_\theta = g$ and for given $x$, the functions $z \mapsto f_z(x)$ and $z \mapsto T(g_z)(x)$ are holomorphic in the interior of $S$ and continuous bounded in $S$. Using the Theorem of holomorphic dependance below the integral, we conclude that $F$ defined on $S$ by

$$F(z) = \int_Y T(f_z)\, g_z\, d\nu$$

satisfies the assumptions of the Phragmen-Lindelöf Lemma. More precisely, $\forall t \in \mathbb{R}$,

$$\|f_{it}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1, \qquad \|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1$$

and

$$\|g_{it}\|_{L^{q_0'}} = \|g\|_{L^{q'}}^{q'/q_0'} = 1, \qquad \|g_{1+it}\|_{L^{q_1'}} = \|g\|_{L^{q'}}^{q'/q_1'} = 1.$$

We conclude using Hölder, the above estimates and our assumption on $T$,

$$\begin{aligned} |F(it)| &\le \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q_0'}} \le M_0 \|f_{it}\|_{L^{p_0}} = M_0, \\ |F(1+it)| &\le \|T(f_{1+it})\|_{L^{q_1}} \|g_{1+it}\|_{L^{q_1'}} \le M_1 \|f_{1+it}\|_{L^{p_1}} = M_1. \end{aligned}$$

The Phragmen-Lindelöf implies

$$\forall x+iy \in S,\ |F(x+iy)| \le M_0^{1-x} M_1^x.$$

Noticing that $F(\theta)$ is the lhs of (1.2) concludes the proof. $\square$

$\blacksquare$

### 1.2.2 Extension to space-time Lebesgue spaces

We extend the above results to space time Lebesgue spaces, the proof is elementary and left to the reader. These functional spaces play a distinguished role in the study of linear and non linear wave equations as we shall see when studying the Schrödinger equation in chapter 5.

Let $E$ be a Banach space, $(X,\mu)$ a measured topological space and $p\in[1,+\infty]$. We define $L^p(X;E)$ as the equivalence set of measurable functions $f$ from $X$ to $E$ such that

$$\|f\|_{L^p(X;E)} \stackrel{\text{def}}{=} \left(\int_X \|f(x)\|_E^p\, d\mu(x)\right)^{\frac{1}{p}} < \infty.$$

Proposition 1.1.1 still holds in the folllowing form, (see for example the appendix of [13]).

**Theoreme 1.2.2** (Space time Lebesgue spaces). *$(L^p(X;E),\|\cdot\|_{L^p(X;E)})$ is a Banach space. Moreover, if $p$ is finite, then there exists an isomorphism from the topological dual of $L^p(X;E)$ onto $L^{p'}(X;E')$.*

We will use the space time Lebesgue spaces only when $X$ is an interval $I$ of $\mathbb{R}$ and $E$ is a Lebesgue space of $\mathbb{R}^d$. The corresponding space will be denoted $L^p(I;L^q(\mathbb{R}^d))$. Riesz-Thorin becomes the following:

**Theoreme 1.2.3** (Riesz-Thorin for space time Lebesgue spaces). *Let $1\leq m_0,m_1,p_0,p_1,q_0,q_1,r_0,r_1\leq\infty$. Let $T$ be a linear operator from $L^{m_0}(I;L^{p_0})+L^{m_1}(I;L^{p_1})$ onto $L^{q_0}(I;L^{r_0})+L^{q_1}(I;L^{r_1})$, which is bounded from $L^{m_0}(I;L^{p_0})$ into $L^{q_0}(I;L^{r_0})$ and from $L^{m_1}(I;L^{p_1})$ into $L^{q_1}(I;L^{r_1})$. Then $\forall\theta\in[0,1]$, $T$ is also bounded from $L^{m_\theta}(I;L^{p_\theta})$ into $L^{q_\theta}(I;L^{r_\theta})$ with*

$$\frac{1}{m_\theta}=\frac{1-\theta}{m_0}+\frac{\theta}{m_1},\quad \frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q_\theta}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1},\quad \frac{1}{r_\theta}=\frac{1-\theta}{r_0}+\frac{\theta}{r_1}\cdot$$

*and moreover*

$$\|T\|_{\mathcal{L}(L^{m_\theta}(I;L^{p_\theta});L^{q_\theta}(I;L^{r_\theta}))}\leq \|T\|^{1-\theta}_{\mathcal{L}(L^{m_0}(I;L^{p_0});L^{q_0}(I;L^{r_0}))}\|T\|^{\theta}_{\mathcal{L}(L^{m_1}(I;L^{p_1});L^{q_1}(I;L^{r_1}))}\cdot$$

## 1.3 Convolution estimates

We present here the classical convolution estimates which appear in many physical models and various analysis problems. It is also a central tool for Fourier analysis. We extend the classical Young inequalities to the Hardy-Littlewood-Sobolev inequalities which allows for the treatment of the singular kernels of mathematical physics. The proof relies onto the atomic decomposition of $L^p$ space which is another intrusion into harmonic analysis techniques.

### 1.3.1 Convolution estimates

**Definition 1.3.1** (Convolution). *Let $\phi\in\mathcal{C}_c(\mathbb{R}^d)$ and $f\in L^1_{\text{loc}}(\mathbb{R}^d)$, then*

$$\phi\star f(x)=\int_{\mathbb{R}^d}\phi(x-y)f(y)dy. \tag{1.3}$$

The standard convolution estimate is Young's inequality.

**Lemma (Inégalité de Young).** *Let* $(p,q,r)\in[1,\infty]^3$ *with*

$$\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}, \tag{1.4}$$

*then the bilinear convolution map* (1.3) *extends uniquely as a bilinear continuous map with*

$$\forall (f,g)\in L^p\times L^q,\quad f\star g\in L^r\quad et\quad \|f\star g\|_{L^r}\leq \|f\|_{L^p}\|g\|_{L^q}. \tag{1.5}$$

*Proof of Lemma 1.3.1.* It suffices to prove (1.5) for $(f,g)\in\mathcal{C}_c\times\mathcal{C}_c$, and then the extension claim follows by extending the obtained continuous bilinear form on the dense subset $\mathcal{C}_c\times\mathcal{C}_c$ of $L^p\times L^q$. The proof we propose uses only Hölder, but we could as well rely on complex interpolation (see exercice 1.3). The case $r=+\infty$ is Hölder, assume now that $r$ is finite, and observe that we may without loss of generality assume $f,g\geq 0$ and $\|f\|_{L^p}=\|g\|_{L^q}=1$. Then $\forall\theta\in]0,1[$,

$$(f\star g)(x)=\int_{\mathbb{R}^d} f^\theta(x-y)g^{1-\theta}(y)f^{1-\theta}(x-y)g^\theta(y)\,dy.$$

Hölder implies: $\forall s\geq 1$, $\forall\theta\in]0,1[$,

$$(f\star g)^r(x)\leq\left(\int_{\mathbb{R}^d} f^{\theta s}(x-y)g^{(1-\theta)s}(y)\,dy\right)^{\frac{r}{s}}\left(\int_{\mathbb{R}^d} f^{(1-\theta)s'}(x-y)g^{\theta s'}(y)\,dy\right)^{\frac{r}{s'}}.$$

We choose $\theta$ and $s$ such that $\theta s=p$ and $\theta s'=q$. Using (1.4), we obtain

$$\theta=\frac{r}{r+1}\,,\quad s=\frac{p(r+1)}{r}\quad\text{and}\quad s'=\frac{q(r+1)}{r}\,\cdot \tag{1.6}$$

We conclude

$$(f\star g)^r(x)\leq\left(\int_{\mathbb{R}^d} f^{p}(x-y)g^{\frac{p}{r}}(y)\,dy\right)^{\frac{r}{s}}\left(\int_{\mathbb{R}^d} f^{\frac{q}{r}}(x-y)g^{q}(y)\,dy\right)^{\frac{r}{s'}}.$$

Let

$$\alpha=\frac{qr}{p}\quad\text{and}\quad\beta=\frac{pr}{q}\,\cdot$$

Since $r\geq\max\{p,q\}$, both $\alpha$ and $\beta$ are bigger or equal to 1. Using Hölder with $\alpha$ (resp. $\beta$) and the probability $f^p(x-y)\,dy$ (resp. $g^q(y)\,dy$), we obtain

$$(f\star g)^r(x)\leq\left(\int_{\mathbb{R}^d} f^{p}(x-y)g^{q}(y)\,dy\right)^{r\left(\frac{1}{s\alpha}+\frac{1}{s'\beta}\right)}.$$

By definition of $\theta$, $s$, $\alpha$ et $\beta$,

$$r\left(\frac{1}{s\alpha}+\frac{1}{s'\beta}\right)=r\left(\frac{rp}{p(r+1)qr}+\frac{rq}{q(r+1)pr}\right)=\frac{r}{r+1}\left(\frac{1}{q}+\frac{1}{p}\right)=1.$$

Hence

$$(f\star g)^r(x)\leq(f^p\star g^q)(x).$$

and the claim follows by integration. □

**Remark 1.3.1.** *Convolution extend to sequences : let* $(a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}$, *we define* $a\star b$ *by*

$$(a\star b)_n \overset{def}{=} \sum_{m\in\mathbb{Z}} a_m b_{n-m} = \sum_{m\in\mathbb{Z}} a_{n-m} b_m.$$

*For* $p\in[1,\infty[$, *let* $\ell^p$ *be the set of sequences* $(a_n)_{n\in\mathbb{Z}}$ *such that*

$$\|(a_n)_{n\in\mathbb{N}}\|_{\ell^p} \overset{def}{=} \left(\sum_{n\in\mathbb{Z}} |a_n|^p\right)^{\frac{1}{p}}$$

*and* $\ell^\infty$ *be the set of bounded sequences. Then the above proof ensures:* $\forall (p,q,r)\in[1,\infty]^3$ *satisfying* (1.4) *and* $(a,b)\in\ell^p\times\ell^q$, *there holds*

$$a\star b\in\ell^r \quad et \quad \|a\star b\|_{\ell^r}\le\|a\|_{\ell^p}\|b\|_{\ell^q}.$$

The weakness of Young' inequality is that it cannot address the case of singular convolution kernels. For example, the Coulomb potential created by a distribution of mass $(\rho(x), x\in\mathbb{R}^3)$ is given by [2]

$$V = -\frac{1}{4\pi|x|}\star\rho,$$

but the kernel $\frac{1}{|x|}$ does not belong to any $L^p(\mathbb{R}^3)$ space. Howeve, the following holds.

**Theorem** (Hardy-Littlewood-Sobolev inequality)**.** *Let* $\alpha\in]0,d[$ *et* $(p,r)\in]1,\infty[^2$ *such that*

$$\frac{1}{p}+\frac{\alpha}{d} = 1+\frac{1}{r}\cdot \tag{1.7}$$

*Then*

$$\forall f\in L^p(\mathbb{R}^d), \quad \||\cdot|^{-\alpha}\star f\|_{L^r(\mathbb{R}^d)} \le C_{r,p}\|f\|_{L^p(\mathbb{R}^d)}.$$

This result is a special case of more general convolution estimates which *precise* Young's inequality.

**Definition 1.3.2** (weal $L^q$ space)**.** *Let* $q\in[1,+\infty[$, *then the weak* $L^q$ *space noted* $L^q_w$ *is the set of fonctions* $g$ *on* $\mathbb{R}^d$ *Lebesgue mesurable such that*

$$\|g\|^q_{L^q_w} \overset{def}{=} \sup_{\lambda>0}\lambda^q \mid \{|g|>\lambda\} \mid < \infty.$$

*Remark.* $L^q\subset L^q_w$ since

$$\lambda^q \mid \{|g|>\lambda\} \mid \le \int_{|g|>\lambda} |g(x)|^q\,dx \le \|g\|^q_{L^q}. \tag{1.8}$$

However,

$$\frac{1}{|x|^\alpha}\in L^{\frac{d}{\alpha}}_w \tag{1.9}$$

but does not belong to any $L^p$.

We may now state the precised convolution estimates which together with (1.9) immediately imply Hardy-Littlewood-Sobolev.

[2] après normalisation des constantes physiques.

**Theoreme 1.3.1** (Precised convolution estimates)**.** *Let* $(p, q, r) \in ]1, \infty[^3$ *satisfiying* (1.4)*. Then there exists* $C$ *such that for all* $(f, g) \in L^p(\mathbb{R}^d) \times L^q_w(\mathbb{R}^d)$, *there holds*

$$\|f \star g\|_{L^r} \le C\|f\|_{L^p}\|g\|_{L^q_w}.$$

**Remark 1.3.2.** *The standard proof of Hardy-Littlewood-Sobolev relies on the maximal function (see exercice 1.5) but does not allow one to obtain Theorem 1.3.1. Some exponents of the Hardy-Littlewood-Sobolev can also be recovered through Sobolev embedding Theorems, see exercice 4.20.*

The end of this chapter is devoted to the proof of Theorem 1.3.1 which requires the introduction of the atomic decomposition of $L^p$ spaces.

### 1.3.2 Atomic decomposition of $L^p$ spaces

We call *atomic decomposition* of a function $f \in L^p$ (with $1 \le p < +\infty$) a characterization given by the following proposition

**Proposition 1.3.1.** *Let* $(X, \mu)$ *be a measured space and* $1 \le p < +\infty$*. For all* $f \in L^p$ *positive, there exists a sequence* $(c_k)_{k\in\mathbb{Z}}$ *and a sequence of positive bounded functions* $(f_k)_{k\in\mathbb{Z}}$ *(called* atoms*) with support two by two disjoints such that*

$$f = \sum_{k\in\mathbb{Z}} c_k f_k$$

*with*

$$\mu(\text{Supp } f_k) \le 2^{k+1}, \tag{1.10}$$

$$\|f_k\|_{L^\infty} \le 2^{-\frac{k}{p}}, \tag{1.11}$$

$$\frac{1}{2}\|f\|^p_{L^p} \le \sum_{k\in\mathbb{Z}} c_k^p \le 2\|f\|^p_{L^p}. \tag{1.12}$$

*Proof of Proposition 1.3.1.* We need only treat $p = 1$. Indeed, $f \in L^p$ iff $|f|^p \in L^1$ and

$$\|f\|^p_{L^p} = \||f|^p\|_{L^1}.$$

Let then $f \in L^1$ positive. Let $E_\lambda \stackrel{\text{def}}{=} \{f > \lambda\}$. The function $\lambda \mapsto \mu(E_\lambda)$ is non increasing on $\mathbb{R}^+$, and converges to 0 at infinity (from (1.8)). For $k \in \mathbb{Z}$, let

$$\lambda_k \stackrel{\text{def}}{=} \inf\left\{\lambda \ /\mu(E_\lambda) < 2^k\right\}, \quad c_k \stackrel{\text{def}}{=} 2^k\lambda_k \quad \text{and} \quad f_k \stackrel{\text{def}}{=} c_k^{-1}\mathbf{1}_{\{\lambda_{k+1}<f\le\lambda_k\}}f.$$

The sequence $(\lambda_k)_{k\in\mathbb{Z}}$ is non increasing and converges to 0 as $k$ tends to $+\infty$. Moreover, since $E_{\lambda_{k+1}} = \bigcup_{\lambda>\lambda_{k+1}} E_\lambda$, we have $\mu(E_{\lambda_{k+1}}) \le 2^{k+1}$, and hence (1.10) is satisfied. Clearly $\|f_k\|_{L^\infty} \le 2^{-k}$. Express

$$\sum_{k\in\mathbb{Z}} c_k = \sum_{k\in\mathbb{Z}} 2^k\lambda_k = \sum_{k\in\mathbb{Z}} \int_0^\infty 2^k\mathbf{1}_{]0,\lambda_k[}(\lambda)\,d\lambda.$$

From Fubini,

$$\sum_{k\in\mathbb{Z}} c_k = \int_0^\infty \Big(\sum_{k\,/\,\lambda_k>\lambda} 2^k\Big)\,d\lambda.$$

By definition of $(\lambda_k)_{k\in\mathbb{Z}}$, $\lambda<\lambda_k$ implies $\mu(E_\lambda)\geq 2^k$ and hence

$$\sum_{k\in\mathbb{Z}} c_k \leq \int_0^\infty \lambda\biggl(\sum_{k\,/\,2^k\leq\mu(E_\lambda)} 2^k\biggr)d\lambda \leq 2\int_0^\infty \mu(\{f>\lambda\})\,d\lambda.$$

Using Fubbini again

$$\int_0^\infty \mu(E_\lambda)\,d\lambda = \int_0^\infty\int_X \mathbf{1}_{\{f>\lambda\}}\,d\mu(x)\,d\lambda = \int_X\biggl(\int_0^{f(x)} d\lambda\biggr)\,d\mu(x) = \|f\|_{L^1}.$$

To conclude the proof of (1.12), we notice that since the supports of the functions $(f_k)_{k\in\mathbb{Z}}$ are two by two disjoints, there holds:

$$\|f\|_{L^1} = \sum_{k\in\mathbb{Z}} c_k\|f_k\|_{L^1}.$$

Now (1.10) and (1.11) imply

$$\|f_k\|_{L^1}\leq 2 \quad \text{pour tout} \quad k\in\mathbb{Z},$$

and the left inequality in (1.12) is proved. □

*Proof of Theorem 1.3.1.* Let $(f,g)\in L^p\times L^q$ as in the assumptions of the Theorem, and let $h\in L^{r'}$. We may without loss of generality assume that these functions are positive with $\|f\|_{L^p}=\|g\|_{L^q_f}=\|h\|_{L^{r'}}=1$. Let

$$I(f,g,h)\stackrel{\text{def}}{=}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(y)g(x-y)h(x)\,dx\,dy.$$

Let $C_j\stackrel{\text{def}}{=}\{z\in\mathbb{R}^d\,,\ 2^j\leq g(z)<2^{j+1}\}$, then

$$I(f,g,h) \leq 2\sum_{j\in\mathbb{Z}} 2^j I_j(f,h) \quad \text{with} \tag{1.13}$$

$$I_j(f,h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d\times\mathbb{R}^d} f(y)h(x)\mathbf{1}_{C_j}(x-y)\,dx\,dy. \tag{1.14}$$

Since $\|g\|_{L^q_f}=1$, we have $\|\mathbf{1}_{C_j}\|_{L^s}\leq 2^{-j\frac{q}{s}}$ for all $s\in[1,\infty]$. We now apply Young's inequality with $p$, $q$ and $r$ and obtain that $I_j(f,h)\leq 2^{-j}$. This is not sufficient to prove the convergence of the series $\sum 2^j I_j(f,h)$. We therefore introduce the atomic decomposition

$$f=\sum_{k\in\mathbb{Z}} c_k f_k \quad \text{et} \quad h=\sum_{k\in\mathbb{Z}} d_k h_k$$

given by Proposition 1.3.1. We have

$$I_j(f,h)=\sum_{k,\ell} c_k d_\ell I_j(f_k,h_\ell).$$

The gain with this new (apparently more complicated) decomposition is that the atoms $f_k$ and $h_\ell$ all belong to Lebesgue spaces. We may therefore play with the full range of Young and

Hölder inequalities to bound each term of the sum. Let $(a,b)\in[1,\infty]^2$ such that $b\leq a'$, then for all $(k,\ell)\in\mathbb{Z}^2$,

$$I_j(f_k,h_\ell)\leq \|f_k\|_{L^a}\|h_\ell\|_{L^b}\|\mathbf{1}_{C_j}\|_{L^{c'}}\quad\text{with}\quad \frac{1}{a}+\frac{1}{b}=1+\frac{1}{c}\cdot$$

Hence

$$I_j(f_k,h_\ell)\leq 2^{-jq\left(2-\frac{1}{a}-\frac{1}{b}\right)}\|f_k\|_{L^a}\|h_\ell\|_{L^b}.$$

Using Proposition 1.3.1, we obtain

$$2^jI_j(f_k,h_\ell)\leq 2^{jq\left(\frac{1}{q}-2+\frac{1}{a}+\frac{1}{b}\right)}2^{k\left(\frac{1}{a}-\frac{1}{p}\right)}2^{\ell\left(\frac{1}{b}-\frac{1}{r'}\right)}.$$

The condition (1.4) on $(p,q,r)$ implies

$$2^jI_j(f_k,h_\ell)\leq 2^{(jq+k)\left(\frac{1}{a}-\frac{1}{p}\right)}2^{(jq+\ell)\left(\frac{1}{b}-\frac{1}{r'}\right)}. \tag{1.15}$$

Let $\varepsilon\stackrel{\text{def}}{=}\frac{1}{4}\left(\frac{1}{p}-\frac{1}{r}\right)\cdot$ Since $q>1$, the condition (1.4) implies that $p<r$, and hence $\varepsilon>0$. Choose then $a$ and $b$ such that

$$\frac{1}{a}\stackrel{\text{def}}{=}\frac{1}{p}-2\varepsilon\,\mathrm{sg}(jq+k)\ \text{ et }\ \frac{1}{b}\stackrel{\text{def}}{=}\frac{1}{r'}-2\varepsilon\,\mathrm{sg}(jq+\ell)$$

with $\mathrm{sg}\,n=1$ if $n\geq 0$, and $\mathrm{sg}\,n=-1$ if $n<0$. By noticing that $b\leq a'$, the estimate (1.15) becomes thanks to the triange inequality

$$2^jI_j(f_k,h_\ell)\ \leq\ 2^{-2\varepsilon|jq+k|-2\varepsilon|jq+\ell|}\leq 2^{-\varepsilon|jq+k|-\varepsilon|jq+\ell|-\varepsilon|k-\ell|}.$$

Using now remark 1.3.1, we conclude

$$I(f,g,h)\ \leq\ \sum_{j,k,\ell}c_kd_\ell 2^{-\varepsilon|jq+k|-\varepsilon|jq+\ell|-\varepsilon|k-\ell|}\leq\frac{C}{\varepsilon}\sum_{k,\ell}c_kd_\ell2^{-\varepsilon|k-\ell|}\leq\frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{p'}}.$$

Since $r'\leq p'$, we have *a fortiori*,

$$I(f,g,h)\leq\frac{C}{\varepsilon^2}\|(c_k)\|_{\ell^p}\|\|(d_\ell)\|_{\ell^{r'}}.$$

In view of the properties of $(c_k)$ and $(d_\ell)$ given by Proposition 1.3.1, we conclude that there exists $C>0$ such that $I(f,g,h)\leq C$ for all positive functions $f\in L^p$, $g\in L^q$ and $h\in L^{r'}$ with norm 1, and Theorem 1.3.1 is proved. □

■

## 1.4 Exercices

**Exercice 1.1 (Cavalieri's principle).** Let $p\in[1,+\infty[$ and $\mu$ a borelian measure.

$(i)$ Show that for all Borelian function $f$, there holds

$$\|f\|_{L^p}^p=p\int_0^\infty\lambda^{p-1}\mu(|f|>\lambda)\,d\lambda. \tag{1.16}$$

(*ii*) Show more generally that if $\Phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is $\mathcal{C}^1$ non decreasing with $\Phi(0) = 0$, then

$$\int_\Omega \Phi(|f(x)|)\, d\mu(x) = \int_0^{+\infty} \Phi'(\lambda)\, \mu(|f| > \lambda)\, d\lambda.$$

**Exercice 1.2** (Schur's Lemma)**.** Let $K \in \mathbb{R}^+$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function with pp $x \in \mathbb{R}^d$ and pp $y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |k(x, y')|\, dy' \leq K \quad \text{and} \quad \int_{\mathbb{R}^d} |k(x', y)|\, dx' \leq K.$$

Given $f$ integrable on $\mathbb{R}^d$, we define $Tf(x) = \int_{\mathbb{R}^d} k(x, y) f(y)\, dy$.

(*i*) Show that the linear map $T$ is continuous from $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$.

(*ii*) Let $p \in [1, +\infty]$. Show that $T$ extends uniquely as a linear map from $L^p(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ and

$$\forall f \in L^p(\mathbb{R}^d),\ \|Tf\|_{L^p} \leq K \|f\|_{L^p}.$$

Hint: use a similar approach like for the proof of Young or use the Riesz-Thorin interpolation theorem.

**Exercice 1.3.** Propose an elementary proof of Young using the Riesz-Thorin interpolation theorem.

**Exercice 1.4.** Let $T$ be a linear defined on the set $\mathcal{C}_c$ of continuous functions with compact support, and with value into the set of measurable functions. We assume that $T$ commutes with translations ie for $h \in \mathbb{R}^d$ and $f \in \mathcal{C}_c$,

$$T(f \circ \tau_h) = (T(f)) \circ \tau_h.$$

(*i*) Show that for all $f \in \mathcal{C}_c$ and $p \in [1, \infty]$, there holds

$$\lim_{h \to \infty} \|f + f \circ \tau_h\|_{L^p} = 2^{\frac{1}{p}} \|f\|_{L^p}.$$

(*ii*) Conclude that if $T$ can be extended as a bounded operator from $L^p$ to $L^q$, then necessarily $q \geq p$.

**Exercice 1.5** (**Maximal function and Hardy-Littlewood-Sobolev inequality**)**.** Given $f$ a Borelian function on $\mathbb{R}$, we let $\|f\|_{L^1_f} := \sup_{\lambda > 0} \lambda\, \mu(E_\lambda)$ where $\mu$ is the Lebesgue measure, and $E_\lambda := \{|f| > \lambda\}$. We associate to $f$ the maximal funtion $Mf$ defined on $\mathbb{R}$ by

$$Mf(x) := \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy.$$

(*i*) (a) Let $(I_j)_{j \in \{1, \cdots, p\}}$ be a family of open intervals of $\mathbb{R}$. Show that there exists a sub family $(I_{j_k})_{k \in \{1, \cdots, q\}}$ of two by two disjoints intervals such that

$$\sum_{k=1}^q \mu(I_{j_k}) = \mu\Big(\bigcup_{k=1}^q I_{j_k}\Big) \geq \frac{1}{3} \mu\Big(\bigcup_{j=1}^p I_j\Big).$$

Hint : argue by induction after ordering the $I_j$ by decreasing order.

(b) Let $\lambda > 0$ and $K \subset E_\lambda(Mf)$ compact.

i. Show that $K$ can be covered by a finite number of intervals $I_j$ with
$$\int_{I_j} f(x)\,dx > \lambda\mu(I_j).$$

ii. Conclude that $\lambda\,\mu(K) \leq 3\|f\|_{L^1}$.

(c) Show that
$$\forall f \in L^1(\mathbb{R}),\ \|Mf\|_{L^1_f} \leq 3\|f\|_{L^1}.$$

(d) Generalize to higher dimension $d \geq 2$ by defining the maximal function through
$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\,dy.$$

(*ii*) Let $f \geq 0,\ f \in L^p$ for $p \in ]1,+\infty[$. Let $\alpha \in ]0,1[$. We recall
$$\|f\|_{L^p}^p = p\int_0^{+\infty} \lambda^{p-1}\mu\big(\{f > \lambda\}\big)\,d\lambda.$$

(a) Show that $\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$.

(b) Show that $\forall\lambda > 0$,
$$\{Mf > \lambda\} \subset \{Mf^\lambda > (1-\alpha)\lambda\} \quad \text{avec} \quad f^\lambda := (f - \lambda\alpha)\mathbf{1}_{\{f \geq \lambda\alpha\}}.$$

(c) Show that there exists $C$ independant of $d$ such that
$$\mu\big(\{Mf^\lambda > (1-\alpha)\lambda\}\big) \leq \frac{C}{(1-\alpha)\lambda}\|f^\lambda\|_{L^1}.$$

(d) Conclude that there exists $C$ such that for all $p \in ]1,+\infty]$ and $f \in L^p$, there holds
$$\|Mf\|_{L^p} \leq C^{\frac{1}{p}}\frac{p}{p-1}\|f\|_{L^p}.$$

(e) What about $p = 1$ ?

(*iii*) Let us now fix $\alpha \in ]0,d[$ and $1 < p,q < +\infty$ with $1+1/q = 1/p+\alpha/d$. Given $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $R > 0$, let
$$T_1f(x) = \int_{|y|<R} \frac{f(x-y)}{|y|^\alpha}\,dy \ \text{ et } \ T_2f(x) = \int_{|y|\geq R} \frac{f(x-y)}{|y|^\alpha}\,dy.$$
We note $Tf := T_1f + T_2f$.

(a) Check that there exists $C > 0$ such that for all $x \in \mathbb{R}^d$,
$$|T_2f(x)| \leq CR^{d(\frac{1}{p'}-\frac{\alpha}{d})}\|f\|_{L^p}.$$

(b) By decomposing $\int_{|y|<R}$ in $\sum_{k=0}^\infty \int_{2^{-(k+1)}R<|y|\leq 2^{-k}R}$, show that
$$\forall x \in \mathbb{R}^d,\ |T_1f(x)| \leq CR^{d-\alpha}Mf(x).$$

(c) Conclude that there exists $C > 0$ (independant of $f$) such that
$$\forall x \in \mathbb{R}^d,\ |Tf(x)| \leq C(Mf(x))^{p/q}\|f\|_{L^p}^{1-p/q}$$
and obtain the Hardy-Littlewood-Sobolev inequality.

# Chapter 2

# Functional analysis

We present in this chapter basic elements of functional analysis in Hilbert and Banach spaces. The key concept is compactness in infinite dimension which is a central tool throughout this series of lectures. We assume that the reader is familiar with the basic concepts of Hilbertian analysis (projection onto a closed convex set, Riesz representation Theorem, existence of a Hilbertian basis and Parseval identity in the separable case). We refer to [3, 4, 9, 18] for an overview of these notions. A more systematic study of Banach spaces can be found in [5, 33, 36].

## 2.1 Compactness in Banach spaces

We recall in this section the notion of compactness in metric and Banach spaces.

### 2.1.1 Compactness in a metric space

Compactness is a central tool in mathematical physics to derive the existence of a limit for sequences involving an infinite dimensional space. Let us recall the abstract notion of compactness in a topological space.

**Definition 2.1.1** (Compact set). *A topological separated space* $(X, \mathcal{O})$ *is said to be* compact *if one can extract from every covering of* $X$ *by open sets a finite covering.*

In metric spaces, the Bolzano-Weierstrass Theorem yields the sequential characterization of compact sets.

**Theorem** (Bolzano-Weierstrass). *A metric space* $(X, d)$ *is compact iff every sequence of* $X$ *admits a converging subsequence in* $X$*.*

In finite dimension, a set is compact iff it is bounded and closed. This result is false in infinite dimension.

**Theoreme 2.1.1** (Riesz). *Let* $E$ *be a vectorial space, then* $E$ *is finite dimensional iff the unit ball of* $E$ *is compact.*

The correct conclusion is that compactness for the strong norm topology is too much to ask: one must weaken the topology to recover a large set of compact sets.

### 2.1.2 Compact operators

A bounded operator is a continuous linear operator between normed vector spaces. We now define the class of compact operators.

**Definition 2.1.2** (Compact operator)**.** *Let $E, F$ be two Banach spaces. A bounded operator $u \in \mathcal{L}(E;F)$ is compact iff the image by $u$ of the unit ball of $E$ it has compact closure in $F$.*

**Remark 2.1.1.** *By linearity, $\overline{u(A)}$ is a compact of $F$ for every bounded set of $E$. In practice, we often use the following characterization.*

**Lemma 2.1.1** (Sequential formulation of compactness)**.** *Let $E, F$ be two Banach spaces. Then $u \in \mathcal{L}(E;F)$ is compact iff for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ of $E$, we can extract a subsequence such that $u(x_{\phi(n)})$ converges in $F$.*

*Proof of Lemma 2.1.1.* Assume that $u$ is compact and let a bounded sequence $x_n$ of E, $M \stackrel{\text{def}}{=} \sup_n \|x_n\|_E$. Then $u(x_n) \in u(B_E(0,M))$ which closure is compact. Hence we can extract $(u(x_\phi(n)))_{n\in\mathbb{N}}$ converging sequence in F. Let now $A$ a bounded set of $E$ and $y_n \in \overline{u(A)}$, then there exists $z_n \in u(A)$ with

$$\|y_n - z_n\|_F \leq \frac{1}{n}\cdot$$

By assumption, we can extract $z_{\phi(n)}$ converging sequence and then $y_{\phi(n)}$ is also convergent. $\square$

*Example.* A finite rank operator, that is an operator which image is finite dimensional, is always compact.

A canonical way to produce compact operators is the following.

**Proposition 2.1.1** (Compact operators define a closed set)**.** *Let $E, F$ be two Banach spaces. Then the set of compact operators is a closed subset of $\mathcal{L}(E;F)$. Equivalently, a uniform limit of compact operators is compact.*

*Proof of Proposition 2.1.1.* The following proof is canonical of compactness methods and relies on the diagonal extraction argument. Let $u$ be the uniform limit of a sequence of compact operators $u_n$ ie

$$\forall \varepsilon > 0, \quad \exists N(\varepsilon) \ \text{ tel que } \ \forall n \geq N(\varepsilon), \quad \forall x \in E, \quad \|u_n(x) - u(x)\|_F \leq \varepsilon \|x\|_E. \tag{2.1}$$

Let $(x^p)_{p\in\mathbb{N}}$ be a bounded sequence of $E$ with $\|x^p\|_E \leq 1$. Since $u_0$ is compact, we may extract from $(u_0(x^p))_{p\in\mathbb{N}}$ a subsequence $(u_0(x^{\phi_0(p)}))_{p\in\mathbb{N}}$ which converges in $F$. By induction, we construct for all $n \geq 0$ extractions $\phi_0, \cdots, \phi_n, \cdots$ such that $(u_n(x^{\phi_0\circ\cdots\circ\phi_n(p)}))_{p\in\mathbb{N}}$ converges in $F$. We then consider the diagonal sequence defined by

$$\phi(n) = \phi_0 \circ \cdots \circ \phi_n(n)$$

which satisfies by construction

$$\forall n \in \mathbb{N}, \quad \left(u_n(x^{\phi(p)})\right)_{p\in\mathbb{N}} \quad \text{converges in} \quad F. \tag{2.2}$$

Let us now show that the sequence $\left(u(x^{\phi(p)})\right)_{p\in\mathbb{N}}$ is a Cauchy sequence in $F$ which concludes the proof. Indeed, let $\varepsilon > 0$ and $N = N(\varepsilon)$ such that (2.1) holds, then

$$\begin{aligned}
&\|u(x^{\phi(p')}) - u(x^{\phi(p)})\|_F \\
&\quad \leq \|u(x^{\phi(p')}) - u_N(x^{\phi(p')})\|_F + \|u_N(x^{\phi(p')}) - u_N(x^{\phi(p)})\|_F + \|u_N(x^{\phi(p)}) - u(x^{\phi(p)})\|_F \\
&\quad \leq 2\varepsilon + \|u_N(x^{\phi(p')}) - u_N(x^{\phi(p)})\|_F \leq 3\varepsilon
\end{aligned}$$

for $p, p' \geq P(\varepsilon)$ large enough by (2.2) applied to $n = N(\varepsilon)$. $\square$

## 2.2 Weak convergence

The weak topology will allow to us to obtain more compact sets, which is the key to derive the existence of limits in infinite dimension. We present first this notion in separable Hilbert spaces which are the simplest examples of infinite dimensional Banach spaces, and then we briefly present its generalization to Banach spaces. We refer to [5, 9, 36] for a more systematic presentation.

### 2.2.1 Weak convergence in separable Hilbert spaces

Let $(\mathcal{H}, (\cdot|\cdot))$ be a separable (i.e. which admits a dense countable set) Hilbert space. The scalar product is a sesquilinear form which is anti-linear for the second coordinate. Let us recall that a separable Hilbert space always admits a Hilbertian basis $(e_i)_{i\in\mathbb{N}}$.

**Proposition 2.2.1** (Hilberatian basis and Parseval identity)**.** *Let $\mathcal{H}$ be a separable Hilbert space. Then there exists a Hilbertian basis $(e_i)_{i\geq 0}$ such that*

$$x = \sum_{i=0}^{+\infty} x_i e_i \in \mathcal{H} \quad \Longleftrightarrow \quad \sum_{i=0}^{+\infty} |x_i|^2 < +\infty,$$

*and there holds the Parseval identity: :*

$$\|x\|^2 = \sum_{i=0}^{+\infty} |x_i|^2 \quad \textit{avec} \;\; x_i = (x|e_i).$$

*More generally,*

$$\forall x, x' \in \mathcal{H}, \quad \langle x, x' \rangle = \sum_{i\geq 0} \langle x, e_i \rangle \langle x', e_i \rangle \tag{2.3}$$

The weak topology is defined as follows.

**Definition 2.2.1** (Weak topology)**.** *Let $(x^n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathcal{H}$. Let $x \in \mathcal{H}$, then $x^n$ weakly converges to $x$ iff*

$$\forall h \in \mathcal{H}, \; \lim_{n\to\infty} (x^n|h) = (x|h).$$

*We note $x^n \rightharpoonup x$.*

**Proposition 2.2.2** (Properties of weak convergence)**.** *Let $(x^n)_{n\in\mathbb{N}}$ and $(y^n)_{n\in\mathbb{N}}$ be two sequences of elements of $\mathcal{H}$ and $x, y$ deux two elements of $\mathcal{H}$. Then:*
*(i) Strong convergence implies weak convergence:*

$$x^n \to x \Longrightarrow x^n \rightharpoonup x.$$

*(ii) Boundedness*

$$x^n \rightharpoonup x \implies (x^n)_{n\in\mathbb{N}} \textit{ is bounded and } \|x\| \leq \liminf \|x^n\|. \tag{2.4}$$

*(iii) Weak strong convergence:*

$$x^n \to x \;\; \textit{and} \;\; y^n \rightharpoonup y \Longrightarrow \lim_{n\to\infty} (x^n|y^n) = (x|y). \tag{2.5}$$

*Proof of Proposition 2.2.2.* (i) follows from Cauchy-Schwarz:

$$\forall h \in \mathcal{H}, \quad |(x^n|h) - (x|h)| \leq \|h\| \, \|x^n - x\| \to 0 \text{ as } n \to +\infty.$$

(ii) is a direct consequence of the Banach-Steinhaus theorem : let $E, F$ be two Banach spaces and $u_n \in \mathcal{L}(E, F)$ such that

$$\forall x \in E, \quad \sup_n \|u_n(x)\|_F < +\infty,$$

then there exists $C > 0$ such that

$$\forall n, \quad \|u_n\|_{\mathcal{L}(E,F)} \leq C.$$

Applying this to the sequence of linear forms $u_n(h) = \langle x_n, h \rangle$ and noticing that $\|u_n\|_{\mathcal{L}(E,\mathbb{C})} = \|x_n\|$ yields the claim. Moreover

$$\|x^n - x\|^2 = \|x^n\|^2 - 2\langle x, x^n \rangle + \|x\|^2 = \|x^n\|^2 - 2\langle x, x^n - x \rangle - \|x\|^2 \tag{2.6}$$

and hence

$$\|x\|^2 \leq \|x^n\|^2 - 2\langle x, x^n - x \rangle$$

which taking the $\liminf$ yields and using $\langle x, x^n - x \rangle \to 0$ yields:

$$\|x^n\| \leq \liminf \|x^n\|.$$

For (iii):

$$|(x^n|y^n) - (x|y)| \quad \leq \quad |(x^n - x|y^n)| + |(x|y^n - y)| \leq \|x^n - x\| \, \|y^n\| + |(x|y^n - y)|,$$

and since (2.4) ensures that $(y^n)_{n \in \mathbb{N}}$ is bounded:

$$|(x^n|y^n) - (x|y)| \leq \big(\sup_{n \in \mathbb{N}} \|y^n\|\big) \, \|x^n - x\| + |(x|y^n - y)| \to 0 \text{ when } n \to +\infty.$$

□

*Example.* A Hilbertian basis $(e_i)_{i \geq 0}$ is the canonical example of a sequence which converges weakly

$$e_i \rightharpoonup 0 \text{ as } i \to +\infty$$

but not strongly for the norm. Indeed,

$$x = \sum_{i=0}^{+\infty} (x|e_i) \, e_i, \quad \sum_{i=0} |x_i|^2 < +\infty$$

implies $x_i = \langle x, e_i \rangle \to 0$. But since the sequence is orthonormal,

$$\|e_1 - e_j\|^2 = \|e_i\|^2 + \|e_j\|^2 = 2 \text{ for } i \neq j,$$

it is not a Cauchy sequence for the norm.

Weak convergence can be characterized in terms of coordinates in the Hilbertian basis.

**Lemma 2.2.1** (Coordinate characterization of weak convergence)**.** *Let $(x^n)_{n \geq 0}$ be a sequence of $\mathcal{H}$ which coordinate $x_i^n = \langle x^n, e_i \rangle$ in a Hilbertian basis $(e_i)_{i \geq 0}$. Then*

$$x^n \rightharpoonup x^\infty \Leftrightarrow (\forall i \geq 0, \quad \lim_{n \to +\infty} x_i^n = x_i^\infty \text{ and } \sup_n \|x^n\| < +\infty).$$

*Proof of Lemma 2.2.1.* The $\Rightarrow$ implication follows from the definition and (2.4). Conversely, let $h \in \mathcal{H}$ and pick $\varepsilon > 0$. Then by quadratic convergence of $\sum_{i\geq 0} |h_i|^2$, Cauchy Schwarz and the boundedness of $x^n$:

$$\sum_{i\geq I(\varepsilon)} |x_i^n h_i| \leq \left(\sum_{i\geq I(e)} |x_i^n|^2\right)^{\frac{1}{2}} \left(\sum_{i\geq I(\varepsilon)} |h_i|^2\right)^{\frac{1}{2}} \leq \varepsilon \sup_n \|x^n\|^2 \lesssim \varepsilon.$$

Hence there exists $I(\varepsilon)$ such that

$$\forall n \geq 1, \quad \sum_{i\geq I(\varepsilon)} |x_i^n h_i| \leq \varepsilon.$$

Moreover for any $J$:

$$\sum_{i=1}^{J} |x_i^\infty|^2 = \lim_{n\to+\infty} \sum_{i=1}^{J} |x_i^n|^2 \leq \sup_n \|x^n\|^2$$

and hence letting $J \to +\infty$

$$\sum_{i=1}^{+\infty} |x_i^\infty|^2 < +\infty$$

which ensures that $x^\infty \stackrel{\text{def}}{=} \sum_{i=0}^{+\infty} x_i^\infty e_i \in \mathcal{H}$. We then estimate similarly

$$\sum_{i\geq I(\varepsilon)} |x_i^\infty h_i| \leq \varepsilon.$$

Hence for all $n \geq 1$ using (2.3):

$$|\langle x^n - x^\infty, h\rangle| = \left| \sum_{i=0}^{I(\varepsilon)-1} (x_i^n - x_i^\infty) h_i + \sum_{i=I(\varepsilon)}^{+\infty} (x_i^n - x_i^\infty) h_i \right|$$

$$\leq \quad \sum_{i=0}^{I(\varepsilon)-1} |x_i^n - x_i^\infty||h_i| + \sum_{i=I(\varepsilon)}^{+\infty} (|x_i^n + |x_i^\infty|)|h_i| \lesssim \sum_{i=0}^{I(\varepsilon)-1} |x_i^n - x_i^\infty||h_i| + \varepsilon$$

and by the convergence of the $I(\varepsilon)$ first coordinates

$$|\langle x^n - x^\infty, h\rangle| \lesssim \varepsilon \ \text{ for } \ n \geq N(\varepsilon) \ \text{ large enough.}$$

□

We may also quantify the lack of strong convergence through the uniform control of high frequencies.

**Lemma 2.2.2** (Default of strong convergence). *Let $x^n \rightharpoonup x$ in $\mathcal{H}$. Let $(e_i)_{i\in\mathbb{N}}$ an Hilbertian basis and $x_i^n = (x^n|e_i)$. The following conditions are equivalent:*

*(i) Uniform control of high frequencies:*

$$\forall \varepsilon > 0, \quad \exists I(\varepsilon) \ \textit{ such that } \ \forall n \in \mathbb{N}, \quad \sum_{i\geq I(\varepsilon)} |x_i^n|^2 < \varepsilon. \tag{2.7}$$

*(ii) Convergence of the norm:* $\|x^n\|_{\mathcal{H}} \to \|x\|_{\mathcal{H}}$ *quand* $n \to +\infty$.

*(iii) Strong convergence:* $x^n \to x$ *quand* $n \to +\infty$.

*Proof of Lemma 2.2.2.* The fact are $(i)$ and $(ii)$ are equivalent follows from (2.6). (iii) implies (i) is obvious. We now prove that $(i)$ implies $(ii)$. Indeed, pick $\varepsilon > 0$, then for $I(\varepsilon)$ large enough, (2.7) and the convergence $\sum_{i\geq 0}|x_i^\infty|^2 < +\infty$ ensure

$$\forall n \geq 1, \quad \sum_{i\geq I(\varepsilon)} |x_i^n|^2 + |x_i^\infty|^2 \leq \varepsilon.$$

Now on the $I(\varepsilon)$ first coordinates we have $x_i^n \to x_i^\infty$ from Lemma 2.2.1, and hence for $n \geq N(\varepsilon)$ large enough:

$$\left|\sum_{i=0}^{+\infty} |x_i^n|^2 - |x_i^\infty|^2\right| \leq \sum_{i=0}^{I(\varepsilon)} |x_i^n|^2 - |x_i^\infty|^2 + \sum_{i=I(\varepsilon)+1}^{+\infty} |x_i^n|^2 + |x_i^\infty|^2 \leq 2\varepsilon.$$

□

*Example.* Lemma 2.2.2 allows us to recognize a typical compact set in infinite dimension: let $(a_i)_{i\in\mathbb{N}}$ with $\displaystyle\sum_{i=0}^{+\infty}|a_i|^2 < +\infty$, then the Hilbert's cube $\left\{x = \displaystyle\sum_{i=0}^{+\infty} x_i e_i, \quad |x_i| \leq |a_i|\right\}$ is a convex compact subset of $\mathcal{H}$.

Now that the topology has been weakened, we recover the compactness of the unit ball.

**Theoreme 2.2.1** (Weak compactness of the unit ball)**.** *Let $\mathcal{H}$ be a separable Hilbert space, then the unit ball is weakly compact. Equivalently, let $x^n$ be a bounded sequence of $\mathcal{H}$, then we can extract a weakly convergent subsequence.*

*Proof of Theorem 2.2.1.* This a diagonal extraction procedure. Let $M = \sup_{n\in\mathbb{N}} \|x^n\|$, $x_i^n = (x^n|e_i)$, then $i \in \mathbb{N}$ et $n \in \mathbb{N}$,

$$\forall i, n, \quad |x_i^n|^2 \leq \sum_{j=0}^{+\infty} |x_j^n|^2 \leq M^2$$

and hence all sequences $(x_i^n)_{n\in\mathbb{N}}$ are bounded. For $i = 0$, we may extract $(x_0^{\phi_0(n)})_{n\in\mathbb{N}}$ convergent in $\mathbb{C}$:

$$\lim_{n\to+\infty} x_0^{\phi_0(n)} = x_0^\infty \quad \text{as} \quad n \to +\infty.$$

We build by induction on $m$ $\phi_1, \cdots, \phi_m$ such that

$$x_m^{\phi_0\circ\cdots\circ\phi_m(n)} = (x^{\phi_0\circ\cdots\circ\phi_m(n)}|e_m) \longrightarrow x_m^\infty \quad \text{as} \quad n \to +\infty.$$

The diagonal extraction $\phi(n) = \phi_0 \circ \cdots \circ \phi_n(n)$ satisfies by construction

$$\forall m \in \mathbb{N}, \quad x_m^{\phi(n)} \to x_m^\infty \quad \text{as} \quad n \to +\infty. \tag{2.8}$$

Since the coordinates of the bounded sequence $(x^{\phi(n)})_{n\geq 1}$ all converge, it is a weakly converging sequence by Lemma 2.2.1. □

### 2.2.2 Adjoint operator

Let us recall the notion of adjoint in a Hilbert space which is a direct application of Riesz' representation Theorem.

**Definition** (Adjunction)**.** Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. There exists a unique $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_2)$ such that

$$\forall (x,y) \in \mathcal{H}_1 \times \mathcal{H}_2,\ (T(x)|y)_{\mathcal{H}_1} = (x|T^*(y))_{\mathcal{H}_2}.$$

$T^*$ is the adjoint of $T$ and satisfies

$$\|T^*\|_{\mathcal{L}(\mathcal{H}_2;\mathcal{H}_1)} = \|T\|_{\mathcal{L}(\mathcal{H}_1;\mathcal{H}_2)}.$$

**Remark 2.2.1.** *The adjoint can be defined in a more general case, [5, 36]. In the chapter 5, we will use the case when the bounded operator $T : \mathcal{H} \to B$ is defined form $\mathcal{H}$ Hilbert onto $B$ Banach. Letting $B'$ be the topological dual of $B$ de $B$, ie the set of continuous forms on $B$, we notice that for $x' \in B'$, the map*[1]

$$x \longmapsto \langle \overline{x'}, T(x)\rangle_{B'\times B} \overset{def}{=} \overline{x'}(T(x))$$

*is a continuous linear form on $\mathcal{H}$. From Riesz representation Theorem, there exists $T^*(x') \in \mathcal{H}$ such that*

$$\langle \overline{x'}, T(x)\rangle_{B'\times B} = (x \mid T^*(x'))_{\mathcal{H}} \quad \textit{for all} \quad x \in \mathcal{H}.$$

An important consequence of the existence of the adjoint is the equivalence of strongly continuous and weakly continuous linear maps.

**Definition 2.2.2.** *A linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be weakly continuous* faiblement continue *iff $(x_n \rightharpoonup x \Rightarrow T(x_n) \rightharpoonup T(x))$.*

**Proposition 2.2.3** (Weak continuity is equivalent to strong continuity)**.** *A linear map $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ is continuous iff it is weakly continuous.*

*Proof of Proposition 2.2.3.* Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ linear continuous and $x_n \rightharpoonup x$ in $\mathcal{H}_1$. Then $\forall y \in \mathcal{H}_2$,

$$(T(x_n) \mid y)_{\mathcal{H}_2} = (x_n \mid T^*(y))_{\mathcal{H}_1} \longrightarrow_{n\to+\infty} (x \mid T^*(y))_{\mathcal{H}_1} = (T(x) \mid y)_{\mathcal{H}_2}$$

and hence $T$ is weakly continuous. Conversely, let $T$ weakly continuous. If $T$ is not strongly continuous, then $T$ in unbounded and hence there exists $x_n \in \mathcal{H}_1$ with $\|x_n\|_{\mathcal{H}_1} = 1$ such that

$$\|T(x_n)\|_{\mathcal{H}_2} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{2.9}$$

By weak compactness of the unit ball, we may extract $(x_{\phi(n)})_{n\in\mathbb{N}}$ and $x \in \mathcal{H}_1$ with $x_{\phi(n)} \rightharpoonup x$, and hence by assumption $T(x_{\phi(n)}) \rightharpoonup T(x)$. Proposition 2.2.2 implies that $(T(x_{\phi(n)}))_{n\in\mathbb{N}}$ is bounded in $\mathcal{H}_2$ which contradicts (2.9). □

**Remark 2.2.2.** *We will systematically use the following corollary in the sequel. Let two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, $\mathcal{H}_1 \subset \mathcal{H}_2$, such that the embedding* $\mathrm{Id} : (\mathcal{H}_1, \|\cdot\|_{\mathcal{H}_1}) \mapsto (\mathcal{H}_2, \|\cdot\|_{\mathcal{H}_2})$ *is continuous, then*

$$x^n \rightharpoonup x^\infty \quad \textit{in} \quad \mathcal{H}_1 \Rightarrow x^n \rightharpoonup x^\infty \quad \textit{in} \quad \mathcal{H}_2.$$

[1] where the conjuguate is to maintain coherence with respect to the above definition of the adjoint in a Hilbert space.

### 2.2.3 Compact operator in the Hilbertian setting

Weak convergence is a powerful tool to characterize compact operators between Hilbert spaces: they are exactly the bounded operators which transform weakly convergent series into strongly convergent series.

**Proposition 2.2.4** (Characterization of compactness through weak convergence)**.** *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, then $T$ is compact iff*

$$x^n \rightharpoonup x^\infty \quad in \;\; \mathcal{H}_1 \Rightarrow T(x_n) \to T(x) \quad in \;\; \mathcal{H}_2. \tag{2.10}$$

**Remark 2.2.3.** *By linearity, it suffices to check* (2.10) *with* $x^\infty = 0$.

*Proof of Proposition 2.2.4.* $\Rightarrow$ If $T$ is compact, let $x^n \rightharpoonup x^\infty$ in $\mathcal{H}_1$, then $T$ is continuous, hence weakly continuous, and hence $T(x_n) \rightharpoonup T(x)$ in $\mathcal{H}_2$. $x^n$ is weakly convergent and hence bounded in $\mathcal{H}_1$, and since $T$ is compact, we conclude that we may extract $T(x^{\phi(n)})$ strongly convergent in $\mathcal{H}_2$, hence weakly convergent and hence by uniqueness of the weak limit

$$T(x^{\phi(n)}) \rightharpoonup T(x) \;\; \text{in} \;\; \mathcal{H}_2.$$

Hence the sequence $T(x^n)$ takes value in a compact set of $\mathcal{H}_2$, and the only accumulation point is $T(x)$, and hence

$$T(x^n) \rightharpoonup T(x) \;\; \text{in} \;\; \mathcal{H}_2.$$

$\Leftarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence of $\mathcal{H}_1$, then we can extract $x^{\phi(n)}$ weakly convergent in $\mathcal{H}_1$ and hence $T(x^{\phi(n)})$ is strongly convergent in $\mathcal{H}_2$ by assumption, and hence $T$ is compact. $\square$

**Lemma 2.2.3** (Compactness and adjoint)**.** *Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, then $T$ is compact iff $T^*$ is compact.*

*Proof of Lemma 2.2.3.* If $T$ is compact, let $y_n \rightharpoonup y$ in $\mathcal{H}_2$. Since $T^*$ is continuous and hence weakly continuous, there holds $T^*(y_n) \rightharpoonup T^*(y)$ in $\mathcal{H}_1$. Hence since $T$ is compact: $T(T^*(y_n)) \to T(T^*(y))$ in $\mathcal{H}_2$. Moreover

$$\|T^*(y_n)\|^2_{\mathcal{H}_1} \;\; = \;\; (T^*(y_n) \mid T^*(y_n))_{\mathcal{H}_1} = (y_n \mid T(T^*(y_n)))_{\mathcal{H}_2} \to (y \mid T(T^*(y)))_{\mathcal{H}_2} = \|T^*(y)\|^2_{\mathcal{H}_1}$$

since $y_n \rightharpoonup y$ and $T(T^*(y_n)) \to T(T^*(y))$. Hence $\|T^*(y_n)\|_{\mathcal{H}_1} \to \|T^*(y)\|_{\mathcal{H}_1}$ and $T^*(y_n) \rightharpoonup T^*(y)$, which ensures by Proposition 2.2.2 that $T^*(y_n) \to T^*(y)$ and $T^*$ is compact by Proposition 2.2.4. The converse claim follows from $(T^*)^* = T$. $\square$

Finally, we complete Proposition 2.1.1 by showing that all compact operator is the uniform limit of a sequence of finite rank operators.

**Proposition** (Uniform approximation of compact operators)**.** *A bounded operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is compact iff it is the uniform limit of a sequence of finite rank operators.*

*Proof of Proposition 2.2.3.* A finite rank operator is compact, and the uniform limit of a sequence of compact operators is compact by Proposition 2.1.1. Conversely, let $T : \mathcal{H}_1 \to \mathcal{H}_2$ compact. Let $K \stackrel{\text{def}}{=} \overline{T(B(0,1))}$ be a compact of $\mathcal{H}_2$. Let $m \geq 1$ and $\varepsilon_m = \frac{1}{m}$, we may extract from the covering $K \subset \cup_{y\in K} B(y, \varepsilon_m)$ a finite covering . Let $(y_i^m)_{1\leq i\leq N(m)}$ be the center of the corresponding balls of radius $\varepsilon_m$. The set $F_m \stackrel{\text{def}}{=} \text{Vect}(y_1^m, \ldots, y_{N(m)}^m)$ is a finite dimensional vector space, hence closed and convex. Let $P_m$ be the projection onto $F_m$, then

$$\forall y \in K, \quad \text{dist}(y, F_m) = \|y - P_m(y)\|_{\mathcal{H}_2} \leq \varepsilon_m$$

and hence

$$\forall x \in B(0,1), \quad \|T(x) - P_m \circ T(x)\|_{\mathcal{H}_2} \leq \varepsilon_m$$

and hence $T$ is the uniform limit of the sequence of finite rank operators $P_m \circ T$. $\square$

### 2.2.4 Weak$^*$ convergence in Banach spaces

We conclude this chapter by a brief presentation of weak$^*$ convergence in Banach spaces which application to $L^p$ is very useful in non linear problems.

**Definition 2.2.3** (Weak$^*$ convergence). *Let $E$ be a $\mathbb{C}$ or $\mathbb{R}$ Banach space, let $E'$ be its topological dual. We say that a sequence $f_n \in E'$ converges weakly$^*$ to $f \in E'$ iff*

$$\forall x \in E,\ \langle f_n, x\rangle_{E'\times E} \to \langle f, x\rangle_{E'\times E}.$$

*We note $f_n \rightharpoonup f$ faible *.*

Weak$*$ convergence has properties very similar to weak convergence in Hilbert spaces.

**Proposition 2.2.5** (Properties of weak$^*$ convergence). *Let $(x_n)_{n\in\mathbb{N}}$ be sequence of the Banach space $E$ and $(f_n)_{n\in\mathbb{N}}$ a sequence of $E'$. Let $x \in E$ and $f \in E'$. Then:*

*(i)* $f_n \rightharpoonup f \implies (f_n)_{n\in\mathbb{N}}$ *is bounded and* $\|f\|_{E'} \leq \liminf \|f_n\|_{E'}$ *;*

*(ii)* $\lim_{n\to+\infty} \|f_n - f\|_{E'} = 0 \implies f_n \rightharpoonup f$ *weak**;

*(iii)* $x_n \to x$ *and* $f_n \rightharpoonup f$ *faible ** $\Longrightarrow \lim_{n\to+\infty}\langle f_n, x_n\rangle_{E'\times E} = \langle f, x\rangle_{E'\times E}$.

*Proof of Proposition 2.2.5.* Point (i) follows from Banach-Steinhaus. Point (ii) follows from the definition of the norm:

$$|\langle f_n - f, x\rangle_{E'\times E}| \leq \|f_n - f\|_{E'}\|x\|_E.$$

For point (iii), we write

$$\begin{aligned} |\langle f_n, x_n\rangle_{E'\times E} - \langle f, x\rangle_{E'\times E}| &\leq |\langle f_n - f, x\rangle_{E'\times E}| + |\langle f_n, x_n - x\rangle_{E'\times E}| \\ &\leq |\langle f_n - f, x\rangle_{E'\times E}| + \|f_n\|_{E'}\|x_n - x\|_E. \end{aligned}$$

The first term converges to 0 by assumption, and since $f_n$ weakly convergent is bounded, $(\|f_n\|_{E'})_{n\in\mathbb{N}}$ is bounded, and the conclusion follows. $\square$

We then recover the weak$^*$ compactness of the unit ball in a Banach space.

**Theoreme 2.2.2** (Weak$*$ compactness of the unit ball). *Let $E$ be a separable Banach space, then every bounded sequence of $E'$ admits a weakly$^*$ converging subsequence.*

*Proof of Theorem 2.2.2.* Since $E$ is separable, there exists a dense countable family $(e_j)_{j\in\mathbb{N}}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $E'$ bounded by $M$. As in the Hilbertian case, we diagonally extract a subsequence $(f_{\psi(n)})_{n\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$, the sequence $(f_{\psi(n)}(e_j))_{n\in\mathbb{N}}$ is convergent. By linearity, the convergence holds on the vectorial space $V$ spanned by the $e_j$, and the limit function defined is a linear form on $V$. By assumption, the set $\{e_j\,,\, j \in \mathbb{N}\}$ is dense in $E$, and so is $V$. Since $f_n$ is bounded, the limit $f$ is also bounded and hence $f \in V'$. Since $V$ is dense, we conclude that $f$ uniquely extends as a linear form on $E$. It remains to prove that $\forall x \in E$, $\lim_{n\to+\infty}(f_{\psi(n)}(x))_{n\in\mathbb{N}} = f(x)$. Indeed, let $x \in E$ and $\varepsilon > 0$. By density of $V$ in $E$, $\exists y \in E$ with $\|x - y\|_E \leq \varepsilon$, and then:

$$|f_{\psi(n)}(x) - f(x)| \leq |f_{\psi(n)}(x) - f_{\psi(n)}(y)| + |f_{\psi(n)}(y) - f(y)| + |f(y) - \tilde{f}(x)|,$$

and since $(f_{\psi(n)})_{n\in\mathbb{N}}$ et $f$ are $M$-Lipschitz, the conclusion follows. $\square$

### 2.2.5 Duality and weak compactness in $L^p$

We now apply the above concept to $L^p$ spaces. We start with the following dual characterization [2] of $L^p$ functions.

**Lemma 2.2.4** (Dual characterization of $L^p$)**.** *Let $(X,\mu)$ be a measured space with $\mu$ $\sigma$-finie[3], let $f$ be a measurable function and let $p \in [1,+\infty]$. Then*

$$f \in L^p \Leftrightarrow \sup_{\|g\|_{L^{p'}}\leq 1} \int_X |f(x)g(x)| d\mu(x) < \infty, \tag{2.11}$$

*and moreover if $f \in L^p$,*

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}\leq 1} \left| \int_X f(x)g(x)d\mu(x) \right|.$$

*Proof of Lemma 2.2.4.* Assume that $f$ is non zero $\mu$ p.p. Given $\lambda \geq 0$, let $E_\lambda \stackrel{\text{def}}{=} \big(\{|f| \geq \lambda\}\big)$. We start with $p=+\infty$. Fix $\lambda > 0$ such that $\mu\big(E_\lambda\big) > 0$. Let $g_0 \in L^1$, $g_0 \geq 0$ and supported in $\overline{E_\lambda}$, and with integral 1. Let

$$g(x) = \frac{\overline{f}(x)}{|f(x)|} g_0 \quad \text{if} \quad f(x) \neq 0, \quad \text{et} \quad g(x) = 0 \quad \text{otherwise.}$$

Then

$$\int_X f(x)g(x)\,d\mu(x) = \int_X |f(x)| g_0(x)\,d\mu(x) \geq \lambda \int_X g_0\,d\mu(x) \geq \lambda$$

which shows that the quantity (2.11) is infinite if $f$ is not in $L^\infty$, and

$$\|f\|_{L^\infty} \geq \sup_{\|g\|_{L^1}\leq 1} \left| \int_X f(x)g(x)d\mu(x) \right|$$

if $f$ is in $L^\infty$. The converse inequality is obvious, and the case $p=+\infty$ is proved.
Let now $1 \leq p < +\infty$, and consider an increasing sequence $(X_n)_{n\in\mathbb{N}}$ with finite measure which union is $X$. Let

$$f_n = \mathbf{1}_{X_n \cap \{|f|\leq n\}} f, \quad g_n(x) = \frac{\overline{f}_n(x)|f_n(x)|^{p-1}}{|f_n(x)|\,\|f_n\|_{L^p}^{p-1}} \quad \text{if} \quad f_n(x) \neq 0 \quad \text{and} \quad g_n(x) = 0 \quad \text{otherwise.}$$

Then $f_n \in L^1 \cap L^\infty \subset L^p$ and

$$\|g_n\|_{L^{p'}}^{p'} = \frac{1}{\|f_n\|_{L^p}^p} \int_X |f_n(x)|^{(p-1)\frac{p}{p-1}} d\mu(x) = 1.$$

The definitions of $f_n$ and $g_n$ ensures

$$\begin{aligned}
\int_X f(x)\mathbf{1}_{X_n\cap(|f|\leq n)} g_n(x) d\mu(x) &= \int_X f_n(x)g_n(x)d\mu(x) = \left(\int_X |f_n(x)|^p d\mu(x)\right) \|f_n\|_{L^p}^{1-p} \\
&= \|f_n\|_{L^p}
\end{aligned}$$

[2]which in setting of a general Banach space is a consequence of the Hahn-Banach Theorem, cf [5].
[3]i.e. there exist an increasing sequence of borelian sets $(X_n)_{n\in\mathbb{N}}$ with finite measure which union is $X$

and hecne

$$\int_X |f_n(x)|^p d\mu(x) \le \left( \sup_{\|g\|_{L^{p'}}\le 1} \left| \int_X f(x)g(x)d\mu(x) \right| \right)^p .$$

If the rhs is finite, the monotone convergence Theorem applied to the nondecreasing sequence $(|f_n|^p)_{n\in\mathbb{N}}$ yields

$$f \in L^p \quad \text{and} \quad \|f\|_{L^p} \le \sup_{\|g\|_{L^{p'}}\le 1} \left| \int_X f(x)g(x)\, d\mu(x) \right| .$$

Then if $f \in L^p$, let $g(x) = \dfrac{\overline{f}(x)|f(x)|^{p-1}}{|f(x)|\,\|f\|^{p-1}_{L^p}}$, then

$$\|g\|^{p'}_{L^{p'}} = \frac{1}{\|f\|^p_{L^p}} \int_X |f(x)|^{(p-1)\frac{p}{p-1}} d\mu(x) = 1 \quad \text{et} \quad \|f\|_{L^p} = \int_X f(x)g(x)\, d\mu(x),$$

and the claim is proved. □

A fundamental corollary is the computation of the dual of $L^p$.

**Theoreme 2.2.3** (Riesz representation Theorem)**.** *Let $1 \le p < +\infty$ and $p'$ be the conjuguate Hölder exponent. Assume that $\mu$ on $X$ is $\sigma$-finite. Let $\varphi$ a continuous linear form on $L^p$. Then $\exists! u \in L^{p'}$ such that*

$$\forall f \in L^p,\ \langle \varphi, f\rangle \overset{def}{=} \varphi(f) = \int_X uf\, d\mu(x).$$

*and $\|\varphi\|_{(L^p)'} = \|u\|_{L^{p'}}$. Equivalently, the map $T : u \mapsto T_u$ defined for all $u \in L^{p'}$ and $f \in L^p$ by $T_u(f) = \int_X fu\, d\mu(x)$ is an isomorphism from $L^{p'}$ onto $\left(L^p\right)'$.*

*Proof of Theorem 2.2.3.* We established in Lemma 2.2.4 that $T : u \mapsto T_u$ is an isometry of $L^{p'}$ onto $(L^p)'$. The surjectivity is non trivial and relies on geometric properties of the $L^p$ norm (Clarkson's inequality), see [5]. □

*Remark.* The reader familiar with distribution theory knows that Theorem 2.2.3 is false for $p = +\infty$: the *Dirac mass*

$$\langle \delta_0, f\rangle \overset{\text{def}}{=} f(0)$$

is the canonical example of continuous linear form on $\mathcal{C}(\mathbb{R}^d, \|\cdot\|_{L^\infty})$ which cannot be identified with a locally integrable function $u$. Hence the topological dual of $L^\infty$ is strictly bigger than $L^1$: $L^1 \subsetneq (L^\infty)'$.

Riesz representation Theorem implies that for for $1 < p < +\infty$, $L^p$ is a dual : $L^{p'} \simeq (L^p)'$, and is reflexive:

$$(L^p)'' \simeq L^p.$$

Hence $L^p$ is a *reflexive separable* Banach space in this case, and an immediate consequence of Theorem 2.2.2 is the weak$^*$ compactness of the unit ball.

**Corollary 2.2.1** (Weak compactness of the unit ball)**.** *Let $1 < p < +\infty$ and $(f_n)_{n\in\mathbb{N}}$ a bounded sequence of $L^p(A,\mu)$ with $A$ borélian of $\mathbb{R}^d$ and $\mu$ absolutely continuous with respect to the Lebesgue measure. Then there exists a subsequence $(f_{\varphi(n)})_{n\in\mathbb{N}}$ and $f \in L^p(A,\mu)$ such that*

$$\forall g \in L^{p'}(A,\mu), \quad \lim_{n\to+\infty} \int_A f_{\varphi(n)}(x)g(x)\, d\mu(x) = \int_A f(x)g(x)\, d\mu(x).$$

**Remark 2.2.4.** *Since any linear form on the separable space $L^1(A,\mu)$ can be identified to a function $L^\infty(A,\mu)$, Theorem 2.2.2 ensures that the above result remains true for $p=+\infty$ even though $L^\infty(A,\mu)$ is not separable. It is however completely false for $p=1$ (see exercice 2.9).*

## 2.3 Exercices

**Exercice 2.1.** Let $(X,d)$ be a complete metric space. Show that a subset $A$ of $X$ has compact closure iff

$$\forall \varepsilon>0,\ \exists N\in\mathbb{N}^*,\ \exists (x_j)_{1\le j\le N}\in A^N \ /\ A\subset \bigcup_{j=1}^{N} B(x_j,\varepsilon).$$

**Exercice 2.2** (Continuation Theorem)**.**

($i$) Let $(X,d)$, $(Y,\delta)$ be two metric space, $A$ a dense set of $X$ and $f$ a uniformly continuous map from $(A,d)$ to $(Y,\delta)$. Show that if $Y$ is complete, then there exists a continuous map $\widetilde{f}$ from $(X,d)$ into $(Y,\delta)$ such that $\widetilde{f}_{|A}=f$, and $\widetilde{f}$ is uniformly continuous.

($ii$) Let $E$, $F$ be two normed vector spaces, $V$ a dense vectorial subset of $E$ and $L$ a linear map from $V$ into $F$. Assume that $F$ is complete. Show that there exists a continuous linear map $\widetilde{L}$ from $E$ to $F$ such that $\widetilde{L}_{|V}=L$.

**Exercice 2.3.** Let $\mathcal{H}_1$, $\mathcal{H}_2$ Hibert spaces with Hilbertian basis $(e_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$, respectively. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of complex numbers converging to $0$.

($i$) Show that there exists a unique $T\in\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$ with

$$\forall n\in\mathbb{N},\ T(e_n)=\varepsilon_n f_n.$$

($ii$) Show that $T$ is compact.

**Exercice 2.4.** Let a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{C}([0,1];\mathbb{R})$ which converges uniformly to $f$ on $[0,1]$. Let $x\in[0,1]$ and $x_n\to x$. Show that

$$f_n(x_n)\to f(x).$$

Show that $f_n(x)=\sin(nx)$ does not admit any converging subsequence in $\mathcal{C}([0,1[,\|\cdot\|_{L^\infty})$.

**Exercice 2.5.** Let $(X,d)$ be compact metric space. Let $C^\alpha(X;\mathbb{K})$ (with $\mathbb{K}=\mathbb{R}$ ou $\mathbb{C}$) which are Hölderian with index $\alpha\in]0,1]$ from $X$ into $\mathbb{K}$. Let the norm

$$\|f\|_\alpha=\sup_{x\in X}|f(x)|+\sup_{\substack{(x,y)\in X^2\\ x\neq y}}\frac{|f(x)-f(y)|}{d(x,y)^\alpha}.$$

($i$) Show $(C^\alpha(X;\mathbb{K}),\|\cdot\|_\alpha)$ is a Banach space.

($ii$) Show that for all $\alpha$, the embedding of $C^\alpha(X;\mathbb{K})$ into the Banach space of continuous functions from $X$ in $\mathbb{K}$ is compact.

($iii$) Show that given $(f_n)_{n\in\mathbb{N}}$ a bounded sequence of $C^\alpha(X;\mathbb{K})$, there exists $f$ in $C^\alpha(X;K)$ and a subsequence with $f_{\psi(n)}\to f$ in $(C^{\alpha'}(X;K),\|\cdot\|_{\alpha'}$ for all $\alpha'\in]0,\alpha[$.

**Exercice 2.6.** Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Find two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ such that

$$x_n \rightharpoonup x\,,\ y_n \rightharpoonup y \quad \text{et} \quad \lim_{n\to+\infty}(x_n|y_n) \neq (x|y).$$

**Exercice 2.7.** Let $\mathcal{H}$ be a Hilbert space (non necessarily separable). Show that every bounded sequence of $\mathcal{H}$ admits a weakly convergent subsequence.

**Exercice 2.8.** Let $\mathcal{H}$ be a separable Hilbert space. Let $A$ be a closed convex subset of $\mathcal{H}$. Let $\phi : A \to \mathbb{R}$ be a continuous convex function which $\lim_{\|x\|\to+\infty}\phi(x) = +\infty$. Show that $\phi$ is lower bounded and attains its infimum.

**Exercice 2.9.** Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ with integral 1. For $n \geq 1$, we let $\varphi_n(x) = n\varphi(nx)$.

(*i*) Show that the sequence $(\varphi_n)_{n\geq1}$ is bounded in $L^1$, and compute the limit $\int_{\mathbb{R}}\varphi_n f\,dx$ for $f \in L^\infty$, which vanishes p.p. in a neighborhood of the origin.

(*ii*) Does the sequence weakly converge in $L^1$ ?

**Exercice 2.10** (Compactness of the convolution)**.** Let $k \in L^2(\mathbb{R}^d)$ and $T_k(f) = k \star f$. Show that $T_k$ is compact as a linear bounded map from $L^2(\mathbb{R}^d)$ into $L^2(|x| \leq 1)$.

# Chapter 3

# A brief overview of Distributions

We present in this chapter a brief overview of the theory of Distributions which is due to Laurent Schwartz, and which generalizes the notion of function. The key point is the notion of weak derivative which is extremely useful for the study of linear and nonlinear PDE's, and the notion of tempered distributions which allows us to define the continuous Fourier transform of a priori rough objects. A reference book on the subject is [17].

In this chapter, all functions can be considered either real or complex valued. Given a multiindex $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $\phi \in C^\infty(\mathbb{R}^d)$, we define

$$\left| \begin{array}{l} |\alpha| = \alpha_1 + \cdots + \alpha_d \\ \partial^\alpha \phi = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d} \phi. \end{array} \right.$$

## 3.1 Test functions and regularization

### 3.1.1 Test functions

We introduce the space $\mathcal{D}(\mathbb{R}^d)$ of test functions.

**Definition 3.1.1** (Support of a function)**.** *Let $\phi \in \mathcal{C}(\mathbb{R}^d)$, we define*

$$\mathrm{Supp}(\phi) = \overline{\{x, \quad \phi(x) \neq 0\}}.$$

**Definition 3.1.2** (Test functions)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. We let $\mathcal{C}_c(\Omega)$ be the set of continuous functions on $\Omega$ which support is a compact subset of $\Omega$. We let $\mathcal{D}(\Omega)$ be the set of $C^\infty$ functions on $\Omega$ which support is a compact subset of $\Omega$.*

**Remark 3.1.1.** *$\mathcal{D}(\Omega)$ is obviously non empty. For example a $\mathcal{C}^\infty$ function with support the unit ball is given by*

$$\zeta(x) = \left| \begin{array}{ll} e^{-\frac{1}{1-\|x\|^2}} & \text{for } \|x\| < 1 \\ 0 & \text{for } \|x\| \geq 1, \end{array} \right. \tag{3.1}$$

*and since $\Omega$ contains a ball, the claim follows.*

An element of $\mathcal{D}(\Omega)$ has always its support *strictly* included in $\Omega$ in the following sense.

**Lemma 3.1.1** (Strict inclusion of the support)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$ and $K \subset \Omega$ compact. Then there exists $\phi \in \mathcal{D}(\Omega)$ with $0 \leq \phi \leq 1$ and $\phi \equiv 1$ on an open neighborhood of $K$.*

*Proof of Lemma 3.1.1.* $\forall x \in K$, $\exists r_x > 0$ such that $B(x, r_x) \subset \Omega$. Let $\zeta$ be given by (3.1) and $\chi_x(y) = 2e\zeta\left(\frac{x-y}{r_x}\right)$, then $\text{Supp}\chi_x \subset \overline{B(x, r_x)}$, $\chi_x > 0$ and $\chi_x(x) = 2$. Let $U_x = \{y \in \Omega, \chi_x(y) > 1\}$, then $(U_x)_{x\in K}$ is a covering of $K$ compact, and hence $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$. The function $f = \sum_{i=1}^n \chi_{x_i}$ satisfies

$$\left| \begin{array}{l} f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_+) \\ \text{Supp}(f) \subset \Omega \\ f > 1 \quad \text{on} \quad V = U_{x_1} \cup \cdots \cup U_{x_n}. \end{array} \right.$$

Let $I \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ with

$$I(x) = \left| \begin{array}{ll} 0 & \text{for} \quad x \le 0 \\ 1 & \text{for} \quad x \ge 1 \end{array} \right. , \quad I' \ge 0$$

then $\phi = I \circ f$ yields the claim. $\square$

### 3.1.2 Regularization by convolution

**Definition 3.1.3** ($L^1_{\text{loc}}(\Omega)$). *Let $\Omega$ be an open subset of $\mathbb{R}^d$. We define $L^1_{\text{loc}}(\Omega)$ as the set of Lebesgue measurable functions on $\Omega$ which integral over any $K \subset \Omega$ compact is finite.*

We recall the definition of the convolution operation: let $\phi \in \mathcal{C}_c(\mathbb{R}^d)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then

$$\phi \star f(x) = \int_{\mathbb{R}^d} \phi(x-y) f(y) dy$$

which from Lemma 1.3.1 can be extended to all $(f, g) \in L^p \times L^q$.

**Lemma 3.1.2** (Regularity). *Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, then $\phi \star f(x) \in \mathcal{C}^\infty(\mathbb{R}^d)$ with*

$$\partial^\alpha(\phi \star f) = (\partial^\alpha \phi) \star f.$$

*Proof of Lemma 3.1.2.* Let $K = \text{Supp}\phi \subset \{|x| \le R\}$. Let $x_0 \in \mathbb{R}^d$, then

$$\phi(x-y) f(y) \ne 0 \Rightarrow x - y \in K \Rightarrow |y| \le |x| + R.$$

Hence for $|x - x_0| \le 1$,

$$\phi \star f(x) = \int_{\mathbb{R}^d} \phi(x-y) f(y) dy = \int_{|y| \le |x_0| + R + 1} \phi(x-y) f(y) dy$$

and the $\mathcal{C}^\infty$ regularity with the computation of the derivative follow from Lebesgue's Theorem of derivability below the integral. $\square$

**Lemma 3.1.3** (Control of the support). *Let $f \in \mathcal{C}_c(\mathbb{R}^d)$, $g \in \mathcal{C}(\mathbb{R}^d)$, then*

$$\text{Supp}(f \star g) \subset \text{Supp}f + \text{Supp}g \equiv \{x + y, (x, y) \in \text{Supp}f \times \text{Supp}g\}.$$

*Proof.* Let $x \in \mathbb{R}^d$ with $f \star g(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy \ne 0$ then there exists $y \in \mathbb{R}^d$ with $f(x-y) \ne 0$ and $g(y) \ne 0$ and hence $x = x - y + y \in \text{Supp}f + \text{Supp}g$ implies

$$\text{Supp}(f \star g) = \overline{\{x, \quad f \star g(x) \ne 0\}} \subset \overline{\text{Supp}f + \text{Supp}g}.$$

We now claim that $A$ compact and $B$ closed implies $A + B$ closed which yields the claim. Indeed, let $z_n \in A+B$ be a converging sequence, then $z_n = a_n + b_n \to z$ and by compactness of $A$: $a_{\phi(n)} \to a$ implies $b_{\phi(n)} \to z - a \equiv b \in B$ since $B$ is closed, and hence $z = a + b \in A+B$. $\square$

**Definition 3.1.4** (Regularizing sequence)**.** *We call regularizing sequence a family* $(\zeta_\varepsilon)_{\varepsilon>0}$ *with*

$$\left| \begin{array}{l} \zeta_\varepsilon \in \mathcal{D}(\mathbb{R}^d) \\ \mathrm{Supp}\zeta_\varepsilon \subset \{|x| \le r_\varepsilon\}, \quad \lim_{\varepsilon\to 0} r_\varepsilon = 0 \\ \zeta_\varepsilon \ge 0 \\ \int_{\mathbb{R}^d} \zeta_\varepsilon dx = 1. \end{array} \right.$$

**Remark 3.1.2.** *Given* $\zeta \in D(\mathbb{R}^d)$ *with* $\zeta \ge 0$ *and* $\int_{\mathbb{R}^d} \zeta dx = 1$*, then* $\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right)$ *is a regularizing family.*

**Theoreme 3.1.1** (Density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{C}_c(\mathbb{R}^d)$)**.** *Let* $f \in \mathcal{C}_c(\mathbb{R}^d)$ *and* $\zeta_\varepsilon$ *a regularizing sequence, then* $\zeta_\varepsilon \star f \in \mathcal{D}(\mathbb{R}^d)$ *and*

$$\lim_{\varepsilon\to 0} \|\zeta \star f - f\|_{L^\infty} = 0.$$

*Proof of Theorem 3.1.1.* Let $\mathrm{Supp}f \subset \{|x| \le R\}$, then $\mathrm{Supp}(\zeta \star f) \subset \mathrm{Supp}\zeta_e + \mathrm{Supp}f \subset \{|x| \le R + \varepsilon\}$ and hence $\zeta_\varepsilon \star f \in \mathcal{D}(\mathbb{R}^d)$. Then

$$\begin{aligned} |\zeta_\varepsilon \star f(x) - f(x)| &= \left|\int_{\mathbb{R}^d} \zeta_\varepsilon(y)(f(x-y) - f(y))dy\right| \le \sup_{|y|\le r_\varepsilon} |f(x-y) - f(y)| \int_{|y|\le r_\varepsilon} \zeta_\varepsilon(y)dy \\ &\le \sup_{|y|\le r_\varepsilon} |f(x-y) - f(y)| \to 0 \ \text{ as } \ \varepsilon \to 0 \end{aligned}$$

where we used that f is continuous and compactly supported, and hence uniformly continuous. □

**Theoreme 3.1.2** (Density of $\mathcal{D}(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$)**.** *Let* $1 \le p < +\infty$*. Then* $\mathcal{D}(\mathbb{R}^d)$ *is dense in* $L^p(\mathbb{R}^d)$*. Moreoever,*

$$\lim_{\varepsilon\to 0} \|\zeta_\varepsilon \star f - f\|_{L^p} = 0.$$

*Proof of Theorem 3.1.2.* Let $\zeta_\varepsilon$ be a regularizing sequence. Let $f \in L^p(\mathbb{R}^d)$ and $\eta > 0$. Since $\mathcal{C}_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, there exists $\phi \in \mathcal{C}_c(\mathbb{R}^d)$ with $\|f - \phi\|_{L^p} < \eta$. Then by Haussdorf-Young:

$$\begin{aligned} \|f - \zeta_\varepsilon \star f\|_{L^p} &\le \|f - \phi\|_{L^p} + \|\phi - \zeta_\varepsilon \star \phi\|_{L^p} + \|\zeta_\varepsilon \star (f - \phi)\|_{L^p} \\ &\lesssim \eta + \|\phi - \zeta_\varepsilon \star \phi|_{L^p} + \|f - \phi\|_{L^p}\|\zeta_\varepsilon\|_{L^1} \le 2\eta + \|\phi - \zeta_\varepsilon \star \phi|_{L^\infty} \le 3\eta \end{aligned}$$

for $\varepsilon$ sufficiently small where we used that $\phi$ has compact support and Theorem 3.1.1. Now

$$\|f - \zeta_\varepsilon \star \phi\|_{L^p} \lesssim \|f - \zeta \star f\|_{L^p} + \|(f - \phi) \star \zeta_\varepsilon\|_{L^p} \le 2\eta$$

using Haussdorff-Young again, and the density claim follows since $\zeta_\varepsilon \star \phi \in \mathcal{D}(\mathbb{R}^d)$. □

**Remark 3.1.3.** *Using Lemma 3.1.1, one can extend Theorem 3.1.1 and Theorem 3.1.2 to the case of* $\Omega$ *open subset of* $\mathbb{R}^d$ *with* $\mathcal{C}^1$ *boundary.*

## 3.2 The space $\mathcal{D}'(\Omega)$

### 3.2.1 Definition and extension of functions

Let $\Omega$ be an open subset of $\mathbb{R}^d$. Distributions are defined as the dual of $\mathcal{D}(\Omega)$ for a suitable Frechet type topology on $\mathcal{D}(\mathbb{R}^d)$.

**Definition 3.2.1** (Convergence of test functions)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. A sequence $\phi_n \in \mathcal{D}(\Omega)$ is said to converge to $\phi \in \mathcal{D}(\Omega)$ iff there exists $K \subset \Omega$ compact with*

$$\left| \begin{array}{l} \forall n, \quad \mathrm{Supp}\phi_n \subset K \\ \forall \alpha \in \mathbb{N}^d, \quad \lim_{n\to+\infty} \|\partial^\alpha \phi_n - \partial^\alpha \phi\|_{L^\infty(K)} = 0. \end{array} \right.$$

We define the dual associated to this topology as follows.

**Definition 3.2.2** (Distributions $\mathcal{D}'(\Omega)$)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. A distribution $T$ on $\Omega$ is a $\mathbb{R}$ or $\mathbb{C}$ linear form on $\mathcal{D}(\Omega)$ which satisifies the following continuity property: $\forall K \subset \Omega$ compact, $\exists C_K > 0$, $\exists p_K \in \mathbb{N}$ such that $\forall \phi \in \mathcal{D}(\Omega)$ with* $\mathrm{Supp}\phi \subset K$,

$$|\langle T, \phi \rangle| \le C_K \max_{|\alpha| \le p_K} \|\partial^\alpha \phi\|_{L^\infty(K)}.$$

*If the integer $p_K$ can be chosen independently of $K$, then the smallest such $p$ is called the order of $T$. We let $\mathcal{D}'(\Omega)$ be the set of distributions on $\Omega$.*

The link between the above dual definitions and the Frechet space topology is:

**Lemma 3.2.1** (Sequential continuity of $T$)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $T \in \mathcal{D}'(\Omega)$. Then*

$$\phi_n \to \phi \ \text{ in } \ \mathcal{D}(\Omega) \Rightarrow \lim_{n\to+\infty} \langle T, \phi_n \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

*Proof of Lemma 3.2.1 .* By definition, there exists $K \subset \Omega$ compact with $\mathrm{Supp}\phi_n \subset K$, and hence by continuity of $T$:

$$|\langle T, \phi_n - \phi \rangle_{\mathcal{D}',\mathcal{D}}| \le C_K \max_{|\alpha|\le p_K} \|\partial^\alpha(\phi_n - \phi)\|_{L^\infty(K)} \to 0 \ \text{ as } \ n \to +\infty.$$

□

We associate to $f \in L^1_{\mathrm{loc}}(\Omega)$ its distribution

$$\langle T(f), \phi \rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega f(x)\phi(x)dx. \tag{3.2}$$

We claim that this map is injective, which allows us to identify $T(f)$ and $f$ and view distributions as a strict generalization of locally integrable functions.

**Theoreme 3.2.1** (Injection of $L^1_{\mathrm{loc}}(\Omega)$ into $\mathcal{D}'(\Omega)$)**.** *The linear map $f \mapsto T(f)$ from $L^1_{\mathrm{loc}}(\Omega)$ to $\mathcal{D}'(\Omega)$ is injective.*

*Proof of Theorem 3.2.1.* Let $\Omega$ be an open subset of $\mathbb{R}^d$ and $f \in L^1_{\mathrm{loc}}(\Omega)$ such that

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle T(f), \phi \rangle_{\mathcal{D}',\mathcal{D}} = \int_\Omega f(x)\phi(x)dx = 0.$$

Let $x_\in\Omega$ and $r > 0$ such that $B(x_0, 2r) \subset \Omega$. Let $F(x) = \mathbf{1}_{B(x_0,2r)} f(x)$ and $\zeta_\varepsilon$ be a regularizing sequence, then from Theorem 3.1.2:

$$\lim_{\varepsilon\to 0} \|\zeta_\varepsilon \star F - F\|_{L^1(\mathbb{R}^d)} = 0.$$

Let $x \in B(x_0, r)$, we compute for $\varepsilon > 0$ small enough

$$\zeta_\varepsilon \star F(x) = \int_{\mathbb{R}^d} F(y)\zeta_\varepsilon(x-y)dy = \int_{B(x_0,2r)} f(y)\zeta_\varepsilon(x-y)dy = \int_{\mathbb{R}^d} f(y)\zeta_\varepsilon(x-y)dy = 0$$

since $(x-y) \in \mathrm{Supp}\zeta$ implies $|x-y| < r_\varepsilon \ll 1$. Hence $F \equiv 0$ p.p in $B(x_0, 2r)$ and $f \equiv 0$ p.p in $x \in B(x_0, r)$. The conclusion now easily follows. □

**Remark 3.2.1.** *The above map is not surjective. The Dirac mass defined for $x_0 \in \mathbb{R}^d$ by*

$$\langle \delta_{x_0}, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \phi(x_0)$$

*is an element of $\mathcal{D}'(\mathbb{R}^d)$ which is not of the form $T_f$ for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

### 3.2.2 Convergence of distributions

The convergence in the sense of distributions is a sequential convergence.

**Definition 3.2.3** (Convergence in $\mathcal{D}'(\Omega)$)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. We say that*

$$T_n \rightharpoonup T \quad in \quad \mathcal{D}'(\Omega)$$

*iff*

$$\forall \phi \in \mathcal{D}(\Omega), \quad \lim_{n\to+\infty} \langle T, \phi_n \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

The classical Banach-Steinhaus Theorem for Banach spaces extends to Frechet spaces and gives the following uniform boundedness principle.

**Proposition 3.2.1** (Banach Steinhaus for $\mathcal{D}'(\Omega)$)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $K \subset \Omega$ compact. Let $(T_n)_{n\geq 0} \in \mathcal{D}'(\Omega)$ such that*

$$\forall \phi \in \mathcal{D}(\Omega), \quad \mathrm{Supp}\phi \subset K \Rightarrow \langle T_n, \phi \rangle_{\mathcal{D}',\mathcal{D}} \ \ converges\ as \ \ n \to \infty.$$

*Then $\exists p_K \in \mathbb{N}$, $\exists C_K > 0$ such that*

$$\forall \phi \in \mathcal{D}(\Omega), \quad \mathrm{Supp}\phi \subset K \Rightarrow |\langle T_n, \phi \rangle_{\mathcal{D}',\mathcal{D}}| \leq C_K.$$

This automatically implies (like for linear maps between Banach spaces) that the limit of a sequence of distributions is a distribution.

**Corollary 3.2.1** (The limit of a sequence of distributions is a distribution)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $(T_n)_{n\geq 0} \in \mathcal{D}'(\Omega)$ such that*

$$\forall \phi \in \mathcal{D}(\Omega), \langle T_n, \phi \rangle_{\mathcal{D}',\mathcal{D}} \ \ converges\ as \ \ n \to \infty.$$

*Then the linear form*

$$\langle T, \phi \rangle_{\mathcal{D}',\mathcal{D}} \equiv \lim_{n\to+\infty} \langle T_n, \phi \rangle_{\mathcal{D}',\mathcal{D}}$$

*is an element of $\mathcal{D}'(\Omega)$.*

### 3.2.3 Operations on distributions

We now define a number of canonical operations on distributions. The most important one is the notion of derivative in the sense of distributions.

**Definition 3.2.4** (Derivation in $\mathcal{D}'(\Omega)$)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $T \in \mathcal{D}'(\Omega)$ and $j \in \{1, \ldots, d\}$, we define the partial derivative of $T$ along $x_j$ by*

$$\langle \partial_{x_j} T, \phi \rangle_{\mathcal{D}',\mathcal{D}} = -\langle T, \partial_{x_j} \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

*More generally, given $\alpha \in \mathbb{N}^d$, we define*

$$\langle \partial^\alpha T, \phi \rangle_{\mathcal{D}',\mathcal{D}} = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

The fact that the above formula defines an element of $\mathcal{D}'(\Omega)$ is a straightforward consequence of the continuity condition. We claim that the derivation in $\mathcal{D}'(\Omega)$ coincides with the standard notion of partial derivatives for $\mathcal{C}^1$ functions.

**Lemma 3.2.2** (Derivation in $\mathcal{D}'$ for smooth functions)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $f \in \mathcal{C}^1(\Omega)$ and $T(f)$ be the associated distribution given by* (3.2)*, then*

$$\partial_{x_j} T_f = T(\partial_{x_j} f).$$

*Proof of Lemma 3.2.2.* We may without loss of generality assume $j = 1$. Let $\phi \in \mathcal{D}(\Omega)$, then by definition

$$\begin{aligned} & \langle \partial_{x_1} T(f) - T(\partial_{x_1} f), \phi \rangle_{\mathcal{D}',\mathcal{D}} = -\langle T(f), \partial_{x_1}\phi \rangle_{\mathcal{D}',\mathcal{D}} - \langle T(\partial_{x_1} f), \phi \rangle_{\mathcal{D}',\mathcal{D}} \\ = & \ -\int_\Omega f(x)\partial_{x_1}\phi(x)dx - \int \partial_{x_1} f(x)\phi(x)dx = -\int_\Omega \partial_{x_1}(f(x)\phi(x))dx. \end{aligned}$$

The function $g(x) = \phi(x)f(x)$ is $\mathcal{C}^1(\Omega)$ and has compact support in $\Omega$. Hence $\mathrm{Supp} g \subset [a_1, b_1] \times \cdots \times [a_d, b_d]$ and

$$\int_\Omega \partial_{x_1} g(x)dx = \int_{a_1}^{b_1} \cdots \int_{a_d}^{b_d} \partial_{x_1} g(x_1, \dots, x_d)dx_1 \dots dx_d.$$

Let

$$h(x_1) = \int_{a_2}^{b_2} \cdots \int_{a_d}^{b_d} g(x_1, \dots, x_d)dx_2 \dots dx_d,$$

then the support property ensures $h(a_1) = h(b_1) = 0$ and hence integrating by parts:

$$\int_\Omega \partial_{x_1} g(x)dx = \int_{a_1}^{b_1} h'(x_1)dx_1 = h(b_1) - h(a_1) = 0,$$

and the claim is proved. □

Derivation is a continuous operation for the topology of $\mathcal{D}'(\Omega)$.

**Lemma 3.2.3** (Continuity of derivation)**.** *Let $\Omega$ be an open subset of $\mathbb{R}^d$. Let $(T_n)_{n\geq 1} \in \mathcal{D}'(\Omega)$ with $T_n \rightharpoonup T$ in $\mathcal{D}'(\Omega)$, then $\forall i \in \{1, \dots, d\}$, $\partial_{x_i} T_n \rightharpoonup \partial_{x_i} T$ in $\mathcal{D}'(\Omega)$.*

*Proof of Lemma 3.2.3.* Let $\phi \in \mathcal{D}(\Omega)$, then by definition

$$\langle \partial_{x_i} T_n, \phi \rangle_{\mathcal{D}',\mathcal{D}} = -\langle T_n, \partial_{x_i}\phi \rangle_{\mathcal{D}',\mathcal{D}}\phi\rangle_{\mathcal{D}',\mathcal{D}} \to -\langle T, \partial_{x_i}\phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle \partial_{x_i} T, \phi \rangle_{\mathcal{D}',\mathcal{D}}.$$

□

The fundamental weakness of the theory is that we cannot take the product of two distributions (we can through a para product but this relies in a much more refined Fourier analysis). Multiplication by a $\mathcal{C}^\infty$ solution is however canonically defined.

**Definition 3.2.5** (Multiplication by a $\mathcal{C}^\infty$ solution)**.** *Let $a \in \mathcal{C}^\infty(\Omega)$ and $T \in \mathcal{D}'(\Omega)$, then the product $aT$ is defined by*

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle aT, \phi \rangle_{\mathcal{D}',\mathcal{D}} = \langle T, a\phi \rangle_{\mathcal{D}',\mathcal{D}}$$

*and $aT \in \mathcal{D}'(\Omega)$.*

The proof that $aT \in \mathcal{D}'(\Omega)$ is a simple consequence of Leibniz rule and left to the reader.

## 3.3 The Fourier tranform on the Schwartz space

The continuous Fourier transform

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx \tag{3.3}$$

is a fundamental tool for the study of linear and nonlinear waves. We aim at propagating it to distributions, but this requires a restricted class of test functions. Indeed, $\mathcal{D}'(\mathbb{R}^d)$ is not stable by $\mathcal{F}$, but the Schwartz class defined below will be. We introduce the notation for $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$:

$$x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}.$$

### 3.3.1 The Schwartz class

**Definition 3.3.1** (Schwartz class). *We let $\mathcal{S}(\mathbb{R}^d)$ be the class of functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ for which all derivatives decay faster than any polynomial:*

$$\forall p \in \mathbb{N}, \quad N_p(\phi) = \sum_{|\alpha|,|\beta|\leq p} \|x^\alpha \partial^\beta \phi\|_{L^\infty} < +\infty.$$

**Definition 3.3.2** (Polynomial growth). *We say that $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ has polynomial growth iff*

$$\exists n \in \mathbb{N}, \quad \|\frac{f(x)}{\langle x\rangle^n}\|_{L^\infty} < \infty.$$

A natural topology on $\mathcal{S}$ is defined as follows.

**Definition 3.3.3** (Converging sequences in $\mathcal{S}$). *We say $\phi_n \to \phi$ in $\mathcal{S}$ iff*

$$\forall p \in \mathbb{N}, \quad \lim_{n\to+\infty} N_p(\phi_n - \phi) = 0.$$

The density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{S}$ follows from a straightforward localization argument.

**Lemma 3.3.1** (Density of $\mathcal{D}(\mathbb{R}^d)$ in $\mathcal{S}$). *$\forall \phi \in \mathcal{S}$, $\exists \phi_n \in D(\mathbb{R}^d)$ such that $\phi_n \to \phi$ in $\mathcal{S}$.*

This implies in particular using Theorem 3.1.2 that $\mathcal{S}$ is dense in $L^p(\mathbb{R}^d)$, $1 \leq p < +\infty$. We now state the basis stability properties of the Schwartz class.

**Proposition 3.3.1.** *Let $\phi \in \mathcal{S}$, then*

*(i)* $\forall \alpha \in \mathbb{N}^d$, $\partial^\alpha \phi \in \mathcal{S}$.

*(ii) Let $f \in \mathcal{C}^\infty$ such that all its derivatives have polynomial growth, then $f\phi \in \mathcal{S}$.*

*(iii) Let $1 \leq q \leq +\infty$, then $\mathcal{S} \subset L^q$; more precisely, let $(\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d$ with $|\alpha| \leq p$, $|\beta| \leq p$, then*

$$\|x^\alpha \partial^\beta \phi\|_{L^q} \lesssim (N_p(\phi))^{1-\frac{1}{q}} (N_{p+d+1}(\phi))^{\frac{1}{q}}. \tag{3.4}$$

*Proof of Proposition 3.3.1.* The first two points are a straightforward consequence of the definition of the $N_p$ semi norm and the Leibniz rule. We focus onto (3.4). The case $p = +\infty$ is obvious, and we let $1 \leq p < +\infty$ and estimate:

$$\begin{aligned}\int_{\mathbb{R}^d} |\phi(x)|^q dx &\leq \|\phi\|_{L^\infty}^{q-1} \int |\phi(x)|dx \leq \|\phi\|_{L^\infty}^{q-1} \|\langle x\rangle^{d+1}\phi\|_{L^\infty} \int_{\mathbb{R}^d} \frac{dx}{\langle x\rangle^{d+1}} \\ &\lesssim (N_0(\phi))^{q-1} N_{d+1}(\phi)\end{aligned}$$

which yields

$$\|\phi\|_{L^q} \lesssim (N_0(\phi))^{1-\frac{1}{q}} (N_{d+1}(\phi))^{\frac{1}{q}}$$

and immediately implies (3.4). $\square$

### 3.3.2 The Fourier transform on $\mathcal{S}$

We now study the Fourier transform (3.3) on $\mathcal{S}$.

**Lemma 3.3.2.** *Let $\phi \in \mathcal{S}$, then*

*(i) $\mathcal{F}\phi \in \mathcal{S}$ with*

$$\forall p \in \mathbb{N}, \quad N_p(\mathcal{F}\phi) \leq C_p N_{p+d+1}(\phi); \tag{3.5}$$

*(ii) $\mathcal{F}(\partial_{x_j}\phi) = i\xi_j \mathcal{F}\phi$ and $\partial_{\xi_j}\mathcal{F}\phi = \mathcal{F}(-ix_j\phi)$;*

*(iii) Let $a \in \mathbb{R}^d$ and $\tau_a\phi(x) \equiv \phi(x-a)$, then $\mathcal{F}(\tau_\phi)(\xi) = e^{-i\xi\cdot a}\mathcal{F}(\phi)(\xi)$.*

*(iv) $\mathcal{F}(\phi \star \psi) = (\mathcal{F}\phi)(\mathcal{F}\psi)$.*

*Proof of Lemma 3.3.2.* Since $\phi \in \mathcal{S} \subset L^1$, the fact that $\mathcal{F}\phi \in \mathcal{C}^1(\mathbb{R}^d)$ follows from Lebesgue's Theorem of derivability below the integral sign, and

$$\begin{aligned}
\partial_{\xi_j}\mathcal{F}(\phi)(\xi) &= \partial_{\xi_j}\left(\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx\right) = \int_{\mathbb{R}^d} f(x)\partial_{\xi_j}\left(e^{-ix\cdot\xi}\right)dx = -i\int_{\mathbb{R}^d} f(x)x_j e^{-ix\cdot\xi}dx \\
&= -i\mathcal{F}(x_j\phi)(\xi).
\end{aligned}$$

An elementary induction argument using that $(\phi \in \mathcal{S}) \Rightarrow \left(\forall p \in \mathbb{N}, \langle x\rangle^p\phi \in L^1\right)$ ensures that $\mathcal{F}\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$. We compute integrating by parts in $\mathbb{R}$ using the decay of $\phi$ at $\infty$:

$$\begin{aligned}
\mathcal{F}(\partial_{x_1}\phi)(\xi) &= \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\partial_{x_1}\phi(x)dx = \int_{\mathbb{R}^{d-1}} dx_2\dots dx_d\int_{\mathbb{R}}\partial_{x_1}\phi(x)e^{-ix\cdot\xi}dx_1 \\
&= -\int_{\mathbb{R}^{d-1}} dx_2\dots dx_d\int\partial_{x_1}\left(e^{-ix\cdot\xi}\right)\phi dx_1 = \int_{\mathbb{R}^{d-1}} dx_2\dots dx_d\int i\xi_1\phi e^{-ix\cdot\xi}dx_1 \\
&= i\xi_1\mathcal{F}(\phi)(\xi).
\end{aligned}$$

Then

$$\mathcal{F}(\tau_a\phi)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\phi(x-a)dx = \int_{\mathbb{R}^d} e^{-i(x+a)\cdot\xi}\phi(x)dx = e^{-ia\cdot\xi}\mathcal{F}(\phi)(\xi).$$

It remains to prove (3.5). We compute thanks to the above formulas:

$$\begin{aligned}
|\xi^\alpha\partial^\beta\mathcal{F}(\phi)| &= |\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}\partial_{x_1}^{\beta_1}\dots\partial_{x_d}^{\beta_d}\mathcal{F}(\phi)| = \left|\xi_1^{\alpha_1}\dots\xi_d^{\alpha_d}\mathcal{F}\left(x_1^{\beta_1}\dots x_d^{\beta_d}\dots\phi\right)\right| \\
&= \left|\mathcal{F}\left(\partial^\alpha(x^\beta\phi)\right)\right|
\end{aligned}$$

and hence using (3.4):

$$\|\xi^\alpha\partial^\beta\mathcal{F}(\phi)\|_{L^\infty} = \left\|\mathcal{F}\left(\partial^\alpha(x^\beta\phi)\right)\right\|_{L^\infty} \leq \left\|\partial^\alpha(x^\beta\phi)\right\|_{L^1} \lesssim N_{p+d+1}(\phi)$$

and (3.5) is proved. Finally, by Fubbini:

$$\begin{aligned}
\mathcal{F}(\phi\star\psi)(\xi) &= \int_{\mathbb{R}^d}\left(\int\phi(x-y)\psi(y)dy\right)e^{-ix\cdot\xi}dx = \int_{\mathbb{R}^d}\left(\int\phi(x-y)\psi(y)dy\right)e^{-i(x-y)\cdot\xi}e^{-iy\cdot\xi}dx \\
&= \int_{\mathbb{R}^d}\psi(y)e^{-iy\cdot\xi}\left(\int_{\mathbb{R}^d}\phi(x-y)e^{-i(x-y)\cdot\xi}dx\right) = \int_{\mathbb{R}^d}\psi(y)e^{-iy\cdot\xi}\left(\int_{\mathbb{R}^d}\phi(z)e^{-iz\cdot\xi}dz\right) \\
&= \mathcal{F}\phi(\xi)\mathcal{F}\psi(\xi).
\end{aligned}$$

$\square$

The amazing and fundamental feature of the Fourier transform is that it is almost an involution.

**Proposition 3.3.2** (Inversion formula)**.** *The Fourier transform is a* $\mathbb{C}$*-linear isomorphism of* $\mathcal{S}$ *with*

$$\mathcal{F}^{-1}(\phi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\phi(\xi)d\xi. \tag{3.6}$$

*Moreover,* $\mathcal{F}, \mathcal{F}^{-1}$ *are continuous on* $\mathcal{S}$ *in the sense that*

$$\forall p \in \mathbb{N}, \quad \forall \phi \in \mathcal{S}, \quad N_p(\mathcal{F}(\phi)) + N_p(\mathcal{F}^{-1}(\phi)) \le C_p N_{p+d+1}(\phi). \tag{3.7}$$

**Remark 3.3.1.** *It is often more convenient to reexpress* (3.6) *as*

$$\mathcal{F}\mathcal{F}\phi = (2\pi)^d\tilde{\phi}, \quad \tilde{\phi}(x) = \phi(-x). \tag{3.8}$$

In order to prove Proposition 3.3.2, we need the following lemma.

**Lemma 3.3.3** (Fourier transform of Gaussians)**.** *Let* $A \in \mathcal{M}_d(\mathbb{R})$*,* $A = A^* > 0$*. Let the gaussian*

$$G_A(x) = \frac{1}{\sqrt{(2\pi)^d \det A}} e^{-\frac{(A^{-1}x)\cdot x}{2}},$$

*then* $G_A \in \mathcal{S}(\mathbb{R}^d)$ *with*

$$\mathcal{F}(G_A)(\xi) = e^{-\frac{(A\xi)\cdot\xi}{2}}. \tag{3.9}$$

After diagonalizing $A$ in an orthonormal basis, the proof reduces to the one dimensional computation which is straightforward.

*Proof of Proposition 3.3.2.* Let $\varepsilon > 0$ and apply (3.9) with $A = \frac{\mathrm{Id}}{\varepsilon}$ to obtain

$$G_\varepsilon(x) \equiv \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x - \frac{\varepsilon|\xi|^2}{2}}d\xi = \frac{1}{\varepsilon^{\frac{d}{2}}\left(\frac{2\pi}{\varepsilon}\right)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x-\frac{\varepsilon|\xi|^2}{2}}d\xi = \frac{1}{(\sqrt{2\pi})^d}\frac{e^{-\frac{|x|^2}{2\varepsilon}}}{\varepsilon^{\frac{d}{2}}}$$

and hence the sequence $(G_\varepsilon)_{\varepsilon>0}$ is almost a regularizing sequence (up to the compact support property). Then by Fubbini

$$\begin{aligned}
&\int_{\mathbb{R}^d} G_\varepsilon(x-y)\phi(y)dy = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{(2\pi)^d}e^{i\xi\cdot(x-y)-\frac{\varepsilon|\xi|^2}{2}}\phi(y)d\xi dy \\
= \ &\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x-\frac{\varepsilon|\xi|^2}{2}}\left(\int_{\mathbb{R}^d}\phi(y)e^{-i\xi\cdot y}dy\right)d\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x-\frac{\varepsilon|\xi|^2}{2}}\mathcal{F}(\phi)(\xi)d\xi \\
\to \ &\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\xi\cdot x}\mathcal{F}(\phi)(\xi)d\xi \quad \text{as } \varepsilon \to 0
\end{aligned}$$

by Lebesgue's dominated convergence Theorem. On the other hand, using the regularizing sequence structure and changing variables in the integral:

$$\begin{aligned}
\int_{\mathbb{R}^d} G_\varepsilon(x-y)\phi(y)dy &= \frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x-y|^2}{2\varepsilon}}}{\varepsilon^{\frac{d}{2}}}\phi(y)dy = \frac{1}{(\sqrt{2\pi})^d}\int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}}\phi(x-\sqrt{\varepsilon}z)dz \\
&\to \frac{\int_{\mathbb{R}^d} e^{-\frac{|z|^2}{2}}dz}{(\sqrt{2\pi})^d}\phi(x) = \phi(x) \quad \text{as} \quad \varepsilon \to 0
\end{aligned}$$

by Lebesgue's dominated convergence Theorem again, and (3.6) is proved. □

The fundamental Corollary of (3.6) is Plancherel's formula.

**Corollary 3.3.1** (Plancherel's formula)**.** *Let $(\phi, \psi) \in \mathcal{S} \times \mathcal{S}$, then*

$$\int_{\mathbb{R}^d} \phi(x)\overline{\psi(x)}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}d\xi.$$

*In particular, $\mathcal{F}$ as an isomorphism of $\mathcal{S}$ extends uniquely to a continuous isomorphism of $L^2(\mathbb{R}^d)$.*

*Proof of Corollary 3.3.1.* We compute from (3.6) and Fubbini:

$$\begin{aligned} &\int_{\mathbb{R}^d} \phi(x)\overline{\psi(x)}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \phi(x)\overline{\int_{\mathbb{R}^d} \mathcal{F}\psi(\xi)e^{ix\cdot\xi}d\xi}dx \\ = &\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\phi(x)e^{-ix\cdot\xi}dx\right)\overline{\mathcal{F}\psi(\xi)}d\xi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}d\xi. \end{aligned}$$

Taking $\phi = \phi$ ensures that $\mathcal{F}$ is $L^2$ continuous on $\mathcal{S}$ which is a dense subset of the Banach space $L^2$, and the extension claim follows. □

## 3.4 Tempered distributions

We do not know how to define the Fourier transform of a general element of $\mathcal{D}'(\mathbb{R}^d)$, not even of $L^1_{\text{loc}}(\mathbb{R}^d)$. But there is a natural dual for $\mathcal{S}$ on which $\mathcal{F}$ is canonically defined.

### 3.4.1 Definition of $\mathcal{S}'$

**Definition 3.4.1** (Tempered distributions)**.** *A linear form $T$ on $\mathcal{S}$ is a tempered distribution iff $\exists p \in \mathbb{N}$, $\exists C_p > 0$ such that*

$$\forall \phi \in \mathcal{S}, \quad |\langle T, \phi\rangle_{\mathcal{S}',\mathcal{S}}| \lesssim C_p N_p(\phi). \tag{3.10}$$

*We note $\mathcal{S}'$ the set of tempered distributions.*

*Example.* Let $1 \le p \le +\infty$, $\frac{1}{p} + \frac{1}{p}' = 1$, $f \in L^p(\mathbb{R}^d)$ and $\phi \in \mathcal{S}$, then

$$\langle|\langle f, \phi\rangle_{\mathcal{S}',\mathcal{S}}| = \left|\int_{\mathbb{R}^d} f(x)\phi(x)dx\right| \lesssim \|f\|_{L^p}\|\phi\|_{L^{p'}} \le CN_{p'+d+1}(\phi)$$

and hence $L^p \subset \mathcal{S}'$. Any function with polynomial growth also defines a tempered distribution. On the other hand it is easily see that $f(x) = e^x$ is not an element of $\mathcal{S}'(\mathbb{R})$, the growth is too important at $+\infty$.

The following stability properties are straightforward consequences of the continuity property.

**Lemma 3.4.1** (Operations on $\mathcal{S}$)**.** *Let $T \in \mathcal{S}'$, then*

*(i)* $\forall \alpha \in \mathbb{N}^d$, $\partial^\alpha T \in \mathcal{S}$;

*(ii) for all $f$ with polynomial growth, $fT \in \mathcal{S}$.*

Convergence in $\mathcal{S}'$ is defined as follows.

**Definition 3.4.2** (Convergence in $\mathcal{S}'$). *We say a sequence $T_n \rightharpoonup T$ in $\mathcal{S}'$ iff*

$$\forall \phi \in \mathcal{S}, \quad \lim_{n\to+\infty} \langle T_n, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \langle T, \phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

It is easily seen that the derivation and multiplication operation by a function $f$ with polynomial growth are continuous with respect to the above topology:

$$T_n \rightharpoonup T \;\text{ in }\; \mathcal{S}' \Rightarrow \left| \begin{array}{l} \partial^\alpha T_n \rightharpoonup \partial^\alpha T \;\text{ in }\; \mathcal{S}' \\ fT_n \rightharpoonup fT \;\text{ in }\; \mathcal{S}' \end{array} \right.$$

### 3.4.2 The Fourier transform on $\mathcal{S}'$

We will define it by duality starting with the following simple observation.

**Lemma 3.4.2** (Duality formulation). *Let $\phi, \psi \in \mathcal{S}$, then*

$$\langle \mathcal{F}\phi, \psi\rangle_{\mathcal{S}',\mathcal{S}} = \langle \phi, \mathcal{F}\psi\rangle_{\mathcal{S}',\mathcal{S}}.$$

*Proof of Lemma 3.4.2.* We compute from Fubbini:

$$\langle \mathcal{F}\phi, \psi\rangle_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}^d} \mathcal{F}\phi(x)\psi(x)dx = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\phi(y)e^{-ix\cdot y}dy\right)\psi(x)dx$$

$$= \int_{\mathbb{R}^d}\phi(y)\left(\int_{\mathbb{R}^d}\psi(x)e^{-ix\cdot y}dx\right) = \int_{\mathbb{R}^d}\phi(y)\mathcal{F}\psi(y)dy = \langle \phi, \mathcal{F}\psi\rangle_{\mathcal{S}',\mathcal{S}}.$$

□

**Definition 3.4.3** (Fourier transform on $\mathcal{S}'$). *Let $T \in \mathcal{S}'$ we define its Fourier transform by*

$$\forall \phi \in \mathcal{S}, \quad \langle \mathcal{F}T, \phi\rangle_{\mathcal{S}',\mathcal{S}} \equiv \langle T, \mathcal{F}\phi\rangle_{\mathcal{S}',\mathcal{S}} \tag{3.11}$$

*and $\mathcal{F}T \in \mathcal{S}'$.*

Note that the fact that $\mathcal{F}T \in \mathcal{S}'$ follows directly from (3.7), (3.10). The following structural properties of $\mathcal{F}$ on $\mathcal{S}'$ are a direct consequence of the corresponding properties on $\mathcal{S}$ and the dual formula (3.11).

**Lemma 3.4.3** (Properties of $\mathcal{F}$ on $\mathcal{S}'$). *The following holds:*

*(i) Let $T \in \mathcal{S}'$, then $\mathcal{F}(\partial_{x_j}T) = i\xi_j\mathcal{F}T$ and $\mathcal{F}(x_jT) = i\partial_{\xi_j}\mathcal{F}T$;*

*(ii) $T_n \rightharpoonup T$ in $\mathcal{S}'$ implies $\mathcal{F}T_n \rightharpoonup \mathcal{F}T$ in $\mathcal{S}'$;*

*(iii) For $\phi \in \mathcal{S}$, let $\tilde{\phi}(x) = \phi(-x)$, and for $T \in \mathcal{S}'$, let $\langle \tilde{T}, \phi\rangle_{\mathcal{S}',\mathcal{S}} \equiv \langle T, \tilde{\phi}\rangle_{\mathcal{S}',\mathcal{S}}$. Then $\mathcal{F}$ is an isomorphism if $\mathcal{S}'$ with*

$$\forall T \in \mathcal{S}', \quad \mathcal{F}^{-1}T = \frac{1}{(2\pi)^d}\widetilde{\mathcal{F}T}. \tag{3.12}$$

*Proof of Lemma 3.4.3.* . The first two claims are obvious from the definitions. We prove (3.12) which is equivalent to

$$\mathcal{F}\mathcal{F}T = (2\pi)^d\tilde{T}.$$

Indeed from (3.8):

$$\langle \mathcal{F}\mathcal{F}T, \phi\rangle_{\mathcal{S}',\mathcal{S}} = \langle T, \mathcal{F}\mathcal{F}\phi\rangle_{\mathcal{S}',\mathcal{S}} = \langle T, (2\pi)^d\tilde{\phi}\rangle_{\mathcal{S}',\mathcal{S}} = \langle (2\pi)^d\tilde{T}, \phi\rangle_{\mathcal{S}',\mathcal{S}}.$$

□

## 3.5 Exercices

**Exercice 3.1.** Division by $x$ in $\mathcal{D}'(\mathbb{R})$.

($i$) Solve $xT = 0$ in $\mathcal{D}'(\mathbb{R})$. More generally, solve $x^m T = 0$, $m \in \mathbb{N}$, in $\mathcal{D}'(\mathbb{R})$.

($ii$) Given $S \in \mathcal{D}'(\mathbb{R})$, solve $xT = S$ in $\mathcal{D}'(\mathbb{R})$.

**Exercice 3.2.** ODE in $\mathcal{D}'(\mathbb{R})$.

($i$) Let $T \in \mathcal{D}'(\mathbb{R})$ with $T' = 0$ in $\mathcal{D}(\mathbb{R})$. Show that $T$ is a constant.

($ii$) Solve $T' - T = \delta$ in $\mathcal{D}'(\mathbb{R})$.

**Exercice 3.3.** Limit of distributions.

($i$) Show that the linear form on $\mathcal{D}(\mathbb{R})$ given by $\left\langle \mathrm{pv}\left(\frac{1}{x}\right), \phi) \right\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\phi(x)}{x} dx$ belongs to $\mathcal{D}'$.

($ii$) Given $\varepsilon > 0$, let the complex valued function $f_\varepsilon(x) = \frac{1}{x+i\varepsilon}$, compute $\lim_{\varepsilon\to 0} f_\varepsilon$ in $\mathcal{D}'(\mathbb{R})$.

**Exercice 3.4.** Derivative and translations. Let $\phi \in \mathcal{D}(\mathbb{R})$, $h \in \mathbb{R}$, we define the translation operation by $\tau_h\phi(x) = \phi(x+h)$. Let $T \in \mathcal{D}'(\mathbb{R})$, we define the translation operation by $\langle \tau_h T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \tau_{-h}\phi\rangle_{\mathcal{D}',\mathcal{D}}$. Show that

$$\lim_{h\to 0} \frac{\tau_h T - T}{h} = T' \text{ in } \mathcal{D}'(\mathbb{R}).$$

**Exercice 3.5.** Computing derivatives in $\mathcal{D}'(\mathbb{R}^d)$.

($i$) Let the Heaviside function be $H(x) = \mathbf{1}_{x>0}$. Let $\tilde{H}(x_1, ..., x_N) = H(x_1)...H(x_N)$ and $\alpha = (1, ...1)$. Show that $\partial^\alpha \tilde{H} = \delta_0$.

($ii$) Show that the linear form on $\mathcal{D}(\mathbb{R}^2)$ given by $\langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}} \phi(x, x)dx$ defines an element of $\mathcal{D}'(\mathbb{R}^2)$. Compute $\partial_x T + \partial_y T$.

**Exercice 3.6** ( Distributions with support a singleton)**.** Let $T \in \mathcal{D}'(\mathbb{R})$ with finite order $p \in \mathbb{N}$ such that

$$\forall \phi \in \mathcal{D}(\mathbb{R}\backslash\{0\}), \quad \langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = 0,$$

we want to show that $T = \sum_{i=0}^p c_i \frac{d^i}{dx^i} \delta_{x=0}$.

($i$) Let $\chi \in \mathcal{D}(\mathbb{R})$ with $\chi(x) = 1$ for $|x| \leq 1$ and $\mathrm{Supp}(\chi) \subset [-2, 2]$. Let $\chi_\varepsilon(x) = \chi\left(\frac{x}{\varepsilon}\right)$. Let $\phi \in \mathcal{D}(\mathbb{R})$, show that $\langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \chi_\varepsilon\phi\rangle_{\mathcal{D}',\mathcal{D}}$.

($ii$) Assume $\frac{d^i\phi}{dx^i} = 0$ for $0 \leq i \leq p$. Show that $\lim_{\varepsilon\to 0}\langle T, \chi_\varepsilon\phi\rangle_{\mathcal{D}',\mathcal{D}} = 0$ and conclude.

($iii$) Extend the result to $\mathcal{D}'(\mathbb{R}^d)$.

**Exercice 3.7.** Fundamental solution of the Laplacian.

($i$) Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with radial symmetry (ie $\phi(x) \equiv \phi(r)$ with $r = \sqrt{\sum_{i=1}^d x_i^2}$.) Show that $\Delta\phi = \frac{d^2\phi}{dr^2} + \frac{d-1}{r}\frac{d\phi}{dr}$.

(*ii*) Let $x \in \mathbb{R}^d$ and define

$$E_d(x) = \begin{cases} |x|^{-(d-2)} & \text{if } d \geq 3, \\ \ln |x| & \text{if } d = 2. \end{cases}$$

Show that $E_d \in \mathcal{C}^\infty(\mathbb{R}^d \backslash \{0\})$ with $\Delta E_d = 0$ in $\mathcal{D}'(\mathbb{R}^d \backslash \{0\})$.

(*iii*) Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. Show that $\langle \Delta E_d, \phi \rangle = \lim_{\varepsilon \to 0^+} \int_{\|x\| > \varepsilon} E_d \Delta \phi \, dx$.

(*iv*) Let $d = 2, 3$. By transforming the above integral using Green's formula, compute $\Delta E_d$ in $\mathcal{D}'(\mathbb{R}^d)$.

**Exercice 3.8.** Let $1 \leq p \leq 2$. Show that the Fourier transform sends $L^p$ onto $L^{p'}$.

# Chapter 4

# Sobolev spaces

Sobolev spaces provide a natural functional setting to study the equations of mathematical physics. We shall treat in details in the forthcoming lectures two applications: variational methods with applications to the existence and stability of solitary waves, and the resolution of a Cauchy problem for a non linear dispersive equation.

In this chaper, we first present Sobolev spaces $H^s(\mathbb{R}^d)$ wich are build on $L^2$ and hence display a Hilbertian structure. We then study Sobolev injection theorems which are a spectacular application of frequency localization techniques using Fourier analysis. We shall give at the end of the chapter a brief overview of $L^p$ based Sobolev spaces which are Banach spaces.

In all the chapter, the key word is *compactness* of the Sobolev embeddings.

## 4.1 Sobolev spaces $H^s(\mathbb{R}^d)$

We introduce Sobolev spaces $H^s(\mathbb{R}^d)$ through Fourier analysis. The case of a general domain $\Omega \subset \mathbb{R}^d$ is briefly discussed at the end of the chapter.

### 4.1.1 Hilbertian structure

**Definition** (Sobolev space $H^s(\mathbb{R}^d)$)**.** Let $s \in \mathbb{R}$. We say that a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the Sobolev space $H^s(\mathbb{R}^d)$ iff

$$\widehat{u} \in L^2(\mathbb{R}^d; (1+|\xi|^2)^s \, d\xi).$$

We then let

$$\|u\|_{H^s} = \left( \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \tag{4.1}$$

If $u \in \mathcal{S}'$, then $\widehat{u} \in \mathcal{S}'$, and since $(1+|\xi|^2)^s$ has polynomial growth, $(1+|\xi|^2)^s \widehat{u} \in \mathcal{S}'$. Being in $H^s$ thus means that this distribution is also in $L^2$ and we observe the following elementary lemma.

**Lemma 4.1.1** (Characterization of $L^2(\mathbb{R}^d)$ in $\mathcal{D}'(\mathbb{R}^d)$)**.** *A distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ belongs to $L^2(\mathbb{R}^d)$ iff*

$$\exists C > 0 \quad \textit{such that} \quad \forall \phi \in \mathcal{D}(\Omega), \quad |\langle T, \phi \rangle_{\mathcal{D}',\mathcal{D}}| \leq C \|\phi\|_{L^2}. \tag{4.2}$$

*Proof of Lemma 4.1.1.* If $T = T(f)$ with $f \in\in L^2(\mathbb{R}^d)$, then this is Cauchy Schwarz:

$$|\langle T(f), \phi\rangle_{\mathcal{D}',\mathcal{D}}| = \left| \int_{\mathbb{R}^d} f(x)\phi(x)dx \right| \lesssim \|f\|_{L^2}\|\phi\|_{L^2}.$$

Conversely, (4.2) means that the linear form $L(\phi) = \langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}}$ is continuous on $\mathcal{D}(\mathbb{R}^d)$ which is a dense subset of $L^2$, and hence it can be uniquely extended to $L^2$. But then by Riesz representation Theorem on the Hilbert space $L^2$, there exists $f \in L^2$ such that

$$\forall g \in L^2, \quad L(g) = \int_{\mathbb{R}^d} f(x)g(x)dx$$

and hence in particular

$$\forall \phi \in \mathcal{D}(\Omega), \quad L(\phi) = \langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}^d} f(x)\phi(x)dx = \langle T(f), \phi\rangle_{\mathcal{D}',\mathcal{D}}$$

and hence $T = T(f) \in L^2$. □

We will systematically use in the sequel the *japanese bracket* :

$$\langle \xi \rangle \stackrel{\text{def}}{=} \sqrt{1 + |\xi|^2}.$$

**Proposition 4.1.1.** *Let $s \in \mathbb{R}$, then $(H^s, (\cdot|\cdot)_{H^s})$ with*

$$(u|v)_{H^s} \stackrel{def}{=} \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi$$

*is a Hilbert space.*

*Proof of Proposition 4.1.1.* The vectorial space structure and the fact that (4.1.1) satisfies the axiom of a scalar product is straightforward. We focus onto completeness. Let $u_n \in H^s$ be a Cauchy sequence, then the sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d; \langle\xi\rangle^{2s}\, d\xi)$ which is complete, and hence there exists $\widetilde{u} \in L^2(\mathbb{R}^d; \langle\xi\rangle^{2s}\, d\xi)$ with

$$\lim_{n\to\infty} \|\widehat{u}_n - \widetilde{u}\|_{L^2(\mathbb{R}^d;\langle\xi\rangle^{2s}\, d\xi)} = 0. \tag{4.3}$$

Hence $(\widehat{u}_n)_{n\in\mathbb{N}} \to \widetilde{u}$ in $\mathcal{S}'$. Let $u = \mathcal{F}^{-1}\widetilde{u}$. Since $\mathcal{F}$ is an isomorphism on $\mathcal{S}'$, $u_n \to u$ in $\mathcal{S}'$ and also in $H^s$ from (4.3). □

*Remark.* We are using nothing more than the fact that $\mathcal{F}$ is an isometry from $H^s$ into $L^2(\mathbb{R}^d; \langle\xi\rangle^{2s}\, d\xi)$.

The Sobolev scale measure the decay of the Fourier transform of $u$, and hence the regularity of $u$. The link with classical derivation is the following.

**Proposition 4.1.2** (Integer Sobolev spaces)**.** *Let $m \in \mathbb{N}$, then $H^m(\mathbb{R}^d)$ coincides with the vectorial space of $L^2$ functions which derivative of order at most $m$ in the sense of distributions belong to $L^2$. Moreover,*

$$\widetilde{\|}u\|_{H^m} \stackrel{def}{=} \sqrt{\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^2}^2}$$

*is a Hilbertian norm on $H^m$ which is equivalent to $\|\cdot\|_{H^m}$.*

*Proof of Proposition 4.1.2.* We have

$$\widetilde{\|u\|}^2_{H^m} = \widetilde{(u|u)}_{H^m} \quad \text{with} \quad \widetilde{(u|v)}_{H^m} \stackrel{\text{def}}{=} \sum_{|\alpha|\le m} \int_{\mathbb{R}^d} \partial^\alpha u(x) \overline{\partial^\alpha v(x)}\, dx$$

and hence the norm is derived from a scalar product. Moreover, there exists a constant $C$ such that

$$\forall \xi \in \mathbb{R}^d\,,\ C^{-1} \sum_{|\alpha|\le m} |\xi|^{2|\alpha|} \le \langle \xi \rangle^{2m} \le C \sum_{|\alpha|\le m} |\xi|^{2|\alpha|}. \tag{4.4}$$

Observe that (i)of Lemma 3.4.3 ensures:

$$\forall \alpha \in \mathbb{N}^d, \quad \partial^\alpha u \in L^2 \Longleftrightarrow \xi^\alpha \widehat{u} \in L^2.$$

Hence

$$u \in H^m \Longleftrightarrow \forall |\alpha| \le m\,,\ \partial^\alpha u \in L^2$$

and (4.4) ensures the equivalence of the norms since $\mathcal{F}$ is up to a constant an $L^2$ isometry. $\square$

**Proposition 4.1.3** (Sobolev ladder)**.** *Let* $s \in \mathbb{R}$*.*

*(i)* $\mathcal{D}(\mathbb{R}^d)$ *is dense in* $H^s(\mathbb{R}^d)$*.*

*(ii) let* $s < t$*, then* $H^t \subset H^s$ *and there holds the interpolation inequality:*

$$\forall \theta \in [0,1], \quad \|u\|_{H^{\theta s + (1-\theta)t}} \le \|u\|^\theta_{H^s} \|u\|^{1-\theta}_{H^t}. \tag{4.5}$$

*(iii) Multiplication by* $\phi \in \mathcal{S}$ *is a bounded operator on* $H^s$*.*

*Remark.* Observe that (i) ensures that we could have defined $H^s(\mathbb{R}^d)$ as the smallest Hilbert space complete for the norm (4.1) containing $\mathcal{D}(\mathbb{R}^d)$.

*Proof of Proposition 4.1.3.* For (i), let $u \in H^s$ such that $\forall \phi \in \mathcal{D}(\mathbb{R}^d)$, $(\phi|u)_{H^s} = 0$. Then

$$\forall \phi \in \mathcal{D}(\mathbb{R}^d), \ \int_{\mathbb{R}^d} \widehat{\phi}(\xi) \langle \xi \rangle^{2s} \overline{\widehat{u}(\xi)}\, d\xi = 0.$$

Since $\langle \xi \rangle^s \overline{\widehat{u}(\xi)} \in L^2$ and the Fourier transform and the multiplication by $\langle \xi \rangle^s$ are isomorphisms on $\mathcal{S}$, this implies

$$\forall \phi \in \mathcal{S}(\mathbb{R}^d), \ \int_{\mathbb{R}^d} \phi(\xi) \langle \xi \rangle^s \overline{\widehat{u}(\xi)}\, d\xi = 0$$

and hence by density of $\mathcal{S}$ in $L^2$:

$$\forall \phi \in L^2(\mathbb{R}^d), \ \int_{\mathbb{R}^d} \phi(\xi) \langle \xi \rangle^s \overline{\widehat{u}(\xi)}\, d\xi = 0$$

yields $\langle \xi \rangle^s \overline{\widehat{u}(\xi)} = 0$. This yields (i) using the $\overline{V} = (V^\perp)^\perp = \{0\}^\perp = \mathcal{H}$ in the Hilbert space $\mathcal{H} = H^s$.
(ii) follows from Hölder:

$$\|u\|^2_{H^{\theta s + (1-\theta)t}} = \int \left( \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \right)^\theta \left( \langle \xi \rangle^{2t} |\widehat{u}(\xi)|^2 \right)^{1-\theta} d\xi.$$

(iii) is slightly more delicate and is the price to pay for having a definition on the Fourier side[1] From classical density arguments, we need only prove that

$$\forall u \in \mathcal{S}, \quad \|\phi u\|_{H^s} \le C_\phi \|u\|_{H^s}.$$

From (iii) of Lemma 3.3.2 and (3.6):

$$\widehat{\varphi u} = (2\pi)^{-d} \widehat{\varphi} \star \widehat{u},$$

and hence we need to bound the $L^2$ norm of the function:

$$U(\xi) = (1 + |\xi^2|)^{\frac{s}{2}} \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi - \eta)| \times |\widehat{u}(\eta)| \, d\eta.$$

Let $I_1(\xi) = \{\eta \,/\, 2|\xi - \eta| \le |\eta|\}$ and $I_2(\xi) = \{\eta \,/\, 2|\xi - \eta| > |\eta|\}$, then

$$\begin{aligned} U(\xi) &= U_1(\xi) + U_2(\xi) \quad \text{with} \\ U_j(\xi) &= \langle \xi \rangle^s \int_{I_j(\xi)} |\widehat{\varphi}(\xi - \eta)| \times |\widehat{u}(\eta)| \, d\eta. \end{aligned}$$

Observe that for $\eta \in I_1(\xi)$:

$$\frac{1}{2}|\eta| \le |\xi| \le \frac{3}{2}|\eta|.$$

We conclude that for all $s$, there exists $C$ such that for all $(\xi, \eta)$ such that $\eta \in I_1(\xi)$, there holds

$$\langle \xi \rangle^{2s} \le C \langle \eta \rangle^{2s}.$$

Hence

$$U_1(\xi) \le C \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi - \eta)| \langle \eta \rangle^s |\widehat{u}(\eta)| \, d\eta.$$

Since $\widehat{\varphi}$ belongs to $\mathcal{S}$, in particular $\widehat{\varphi}$ belongs to $L^1$. Hence from Young

$$\|U_1\|_{L^2} \le C \|\widehat{\varphi}\|_{L^1} \|u\|_{H^s}.$$

We now treat $U_2$. For $\eta \in I_2(\xi)$, there holds $|\eta| \le 2|\xi - \eta|$. Hence

$$U_2(\xi) \le \langle \xi \rangle^{|s|} \int_{I_2(\xi)} |\widehat{\varphi}(\xi - \eta)| \langle \eta \rangle^{|s|} \langle \eta \rangle^s |\widehat{u}(\eta)| \, d\eta \le C \int_{\mathbb{R}^d} |\widehat{\varphi}(\xi - \eta)| \langle \xi - \eta \rangle^{2|s|} \langle \eta \rangle^s |\widehat{u}(\eta)| \, d\eta.$$

Since $\widehat{\varphi}$ belongs to $\mathcal{S}$, there exists $C$ such that

$$|\widehat{\varphi}(\zeta)| \le C \langle \zeta \rangle^{-d-1-2|s|}$$

and hence

$$U_2(\xi) \le C \int_{\mathbb{R}^d} \langle \xi - \eta \rangle^{-d-1} \langle \eta \rangle^s |\widehat{u}(\eta)| \, d\eta.$$

Hence $\|U_2\|_{L^2} \le C \|u\|_{H^s}$, and (iii) is proved. $\square$

[1] but there will much more advantages!

### 4.1.2 The dual of $H^s$

Since $H^s$ is a Hilbert space, it is isomporphic to its topological dual $(H^s)'$ via the $H^s$ scalar product, this is Riesz representation Theorem. We now revisit this identification using the "pivot" space $L^2$.

**Proposition 4.1.4** (Dual of $H^s$). *Let $s \in \mathbb{R}$ and $f \in \mathcal{S}'$ such that $\widehat{f} \in L^2_{loc}(\mathbb{R}^d)$. Then $f \in H^{-s}$ iff*

$$M_f \overset{def}{=} \sup_{\substack{\varphi\in\mathcal{S} \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big| < \infty.$$

*Moreover, for $f \in H^{-s}$, the linear form $L_f$ defined on $\mathcal{S}$ by $L_f(\varphi) = \langle f, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}$ can be uniquely extended as linear continuous form on $H^s$ and*

$$\|f\|_{H^{-s}} = (2\pi)^d \sup_{\substack{\varphi\in\mathcal{S} \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big| = (2\pi)^d \sup_{\substack{\varphi\in H^s \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{H^{-s}\times H^s}\big|.$$

*Finally, the map $f \mapsto (2\pi)^d\langle f, \cdot\rangle_{H^{-s}\times H^s}$ is an isometric isomorphism from $H^{-s}$ into $(H^s)'$.*

*Proof of Proposition 4.1.4.* Let $f \in H^{-s}$. Observe that for $\varphi \in \mathcal{S}$,

$$\langle f, \overline{\varphi}\rangle_{\mathcal{S}'\times\mathcal{S}} = (2\pi)^{-d}\langle \widehat{f}, \overline{\widehat{\varphi}}\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Since $\mathcal{S}$ and $\mathcal{S}'$ are stable by multplication by $\langle\cdot\rangle^{\pm s}$,

$$\langle f, \overline{\varphi}\rangle_{\mathcal{S}'\times\mathcal{S}} = (2\pi)^{-d}\langle\langle\cdot\rangle^{-s}\widehat{f}, \langle\cdot\rangle^{s}\overline{\widehat{\varphi}}\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Since $\langle\cdot\rangle^{-s}\widehat{f} \in L^2$, Cauchy-Schwarz and the definition of the Sobolev norm ensure:

$$\big|\langle f, \overline{\varphi}\rangle_{\mathcal{S}'\times\mathcal{S}}\big| \;=\; (2\pi)^{-d}\left|\int \langle\xi\rangle^{-s}\widehat{f}(\xi)\,\langle\xi\rangle^{s}\overline{\widehat{\varphi}(\xi)}\,d\xi\right| \leq (2\pi)^{-d}\|f\|_{H^{-s}}\|\varphi\|_{H^s}$$

and hence

$$(2\pi)^d \sup_{\substack{\varphi\in\mathcal{S} \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big| \leq \|f\|_{H^{-s}}.$$

Hence $L_f$ defined in the statement of the Proposition can be uniquely extended as a linear continuous form on $H^s$ with same norm and restriction on $\mathcal{S}$. Hence

$$\sup_{\substack{\varphi\in H^s \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{H^{-s}\times H^s}\big| = \sup_{\substack{\varphi\in\mathcal{S} \\ \|\varphi\|_{H^s}\leq 1}} \big|\langle f, \varphi\rangle_{\mathcal{S}'\times\mathcal{S}}\big| \leq (2\pi)^{-d}\|f\|_{H^{-s}}.$$

Let now $\varphi$ be defined by

$$\widehat{\varphi}(\xi) \overset{\text{def}}{=} \frac{\langle\xi\rangle^{-2s}\overline{\widehat{f}(\xi)}}{\|f\|_{H^s}},$$

then $\varphi \in H^s$ with norm 1, and

$$\langle f, \varphi\rangle_{H^{-s}\times H^s} = (2\pi)^{-d}\|f\|_{H^{-s}}.$$

Hence the linear form $(2\pi)^d\langle f, \cdot\rangle_{H^{-s}\times H^s}$ is of norm exactly $\|f\|_{H^{-s}}$.
The proof is concluded by showing that the previously defined map is surjective. Let a linear form $L$ continuous on $H^s$, we are reduced to the case $s = 0$ by letting

$$M(\varphi) \overset{\text{def}}{=} L(\langle D\rangle^{-s}\varphi) \quad \text{pour} \quad \varphi \in L^2,$$

where $\langle D\rangle^{-s}$ is the fractional derivation operator

$$\mathcal{F}\big(\langle D\rangle^{-s}\varphi\big) = \langle\cdot\rangle^{-s}\mathcal{F}\varphi. \tag{4.6}$$

Clearly $M$ is a continuous linear form on $L^2$ with norm $\|L\|_{(H^s)'}$. Hence from Riesz, there exists $g \in L^2$ such that $\varphi \in L^2$,

$$M(\varphi) = \int g\,\varphi\,dx.$$

Hence

$$\forall\varphi \in \mathcal{S}, L(\langle D\rangle^{-s}\varphi) = \big\langle \langle D\rangle^{s} g, \langle D\rangle^{-s}\varphi\big\rangle_{\mathcal{S}'\times\mathcal{S}}.$$

Since multiplication by $\langle D\rangle^{-s}$ is an isomorphism in $\mathcal{S}$ (voir exercice 4.2), we conclude

$$L(\psi) = \big\langle \langle D\rangle^{s} g, \psi\big\rangle_{\mathcal{S}'\times\mathcal{S}} \quad \text{for all } \psi \in \mathcal{S}.$$

Since $g \in L^2$, the function $\langle D\rangle^s g$ is in $H^{-s}$. By density, we conclude that $L = \langle\langle D\rangle^s g, \cdot\rangle_{H^{-s}\times H^s}$.
Finally, assume $f \in \mathcal{S}'$ satisfies $\widehat{f} \in L^2_{loc}$ and $M_f$ is finite. Then for all $K > 0$, the fonction $f_K = \mathcal{F}^{-1}\Big[(1_{B(0,K)}\widehat{f}\Big]$ is in $H^{-s}$. Since for all $\varphi \in H^s$, $\mathcal{F}^{-1}(1_{B(0,K)}\widehat{\varphi}) \in H^s$, we easily check

$$\|f_K\|_{H^{-s}} \leq M_f.$$

Using the definition of the $H^{-s}$ norm and the monotone convergence Theroem, we conclude that $f \in H^{-s}$ with norm at most $M_f$. $\square$

## 4.2 The Sobolev injection Theorem

We prove in this chapter the Sobolev injection Theorem. There are two classical proofs of this result[2]: Nirenberg's proof of integration by parts on the space side, see Theorem 4.4.2; and Chemin's proof using a real space interpolation method on the Fourier side. Both have their own interest and in many ways say different things with various applications to geometrical and physical problems.

### 4.2.1 Sobolev injection

**Theoreme 4.2.1** (Sobolev injection in $\mathbb{R}^d$)**.** *Let $s > 0$.*

*(i) If $s > \frac{d}{2}$ then $H^s(\mathbb{R}^d)$ embeds continuously into the space of continuous functions which decay to zero as $|x| \to +\infty$.*

*(ii) If $0 \leq s < \frac{d}{2}$, let the* critical exponent *$p_c$ be given by*

$$-s + \frac{d}{2} = \frac{d}{p_c} \quad i.e. \quad p_c = \frac{2d}{d-2s} \in [2, +\infty[, \tag{4.7}$$

*then for all $p \in [2, p_c]$, $H^s(\mathbb{R}^d)$ embeds continuously into $L^p(\mathbb{R}^d)$:*

$$\exists C_{p,s} > 0 \quad \text{such that} \quad \forall f \in H^s(\mathbb{R}^d), \quad \|f\|_{L^p(\mathbb{R}^d)} \leq C_{p,s}\|f\|_{H^s(\mathbb{R}^d)}. \tag{4.8}$$

*(iii) For $s = \frac{d}{2}$, $H^s(\mathbb{R}^d)$ embeds continuously into $L^p(\mathbb{R}^d)$ for all $2 \leq p < +\infty$.*

[2]on top of Sobolev's proof which is long and delicate, [37].

In fact, we shall need a more precise estimate than (4.8) which is the heart of the analysis.

**Lemma 4.2.1** (Homogeneous Sobolev injection)**.** *Let $0 < s < \frac{d}{2}$ and $p_c$ be given by (4.7). Then there exists $C_s > 0$ such that :*

$$\forall f \in \mathcal{D}(\mathbb{R}^d), \quad \|f\|_{L^{p_c}(\mathbb{R}^d)} \leq C_s \|f\|_{\dot{H}^s(\mathbb{R}^d)} \tag{4.9}$$

*where the homogeneous Sobolev semi-norm is defined by:*

$$\|f\|_{\dot{H}^s} \stackrel{def}{=} \left( \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.$$

*Remark* (The importance of dimension). Let $s = 1$, we conclude that $H^1(\mathbb{R})$ distributions are in fact continuous functions, $H^1(\mathbb{R}^2)$ distributions belong to all $L^p(\mathbb{R}^2)$ with $2 \leq p < +\infty$, and in dimension 3, Theorem 4.2.1 ensures that $H^1(\mathbb{R}^3)$ distributions gain the integrability $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ which is a priori completely non trivial. More precisely, by Plancherel

$$\|f\|_{\dot{H}^1} \eqsim \|\nabla f\|_{L^2},$$

and hence (4.9) yields

$$\forall f \in \mathcal{D}(\mathbb{R}^3), \quad \|f\|_{L^6(\mathbb{R}^3)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^3)}.$$

*Remark* (Scaling invariant homegeneous estimate). A key feature when comparing (4.9) and (4.8) is that (4.9) is scale invariant. In fact, the value of the critical exponent (4.7) can be computed by letting the group of dilations act. Let $f \in \mathcal{D}(\mathbb{R}^d)$ and define the scaled function $f_\lambda(x) \stackrel{\text{def}}{=} f(\lambda x)$ for $\lambda > 0$ then :

$$\forall p \in [1, +\infty], \quad \|f_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}} \|f\|_{L^p}$$

and

$$\left( \int |\xi|^{2s} |\widehat{f_\lambda}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \quad = \quad \left( \lambda^{-2d} \int |\xi|^{2s} |\widehat{f}(\lambda^{-1}\xi)|^2 \, d\xi \right)^{\frac{1}{2}} = \lambda^{-\frac{d}{2}+s} \|f\|_{\dot{H}^s}.$$

Both quantities $\|\cdot\|_{L^p}$ and $\|\cdot\|_{\dot{H}^s}$ scale similarly (and may therefore be compared) iff their scaling match ie $-s + \frac{d}{2} = \frac{d}{p}$ i.e. $p = p_c$.

*Proof of Theorem 4.2.1.* Assume (4.9) which is proved hereafter. We distinguish three cases.
case $s > \frac{d}{2}$. Then $\langle \cdot \rangle^{-s} \in L^2(\mathbb{R}^d)$ and hence by Cauchy-Schwarz:

$$\|\widehat{u}\|_{L^1} \leq \left( \int \langle \xi \rangle^{-2s} \, d\xi \right)^{\frac{1}{2}} \left( \int \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \leq C \|u\|_{H^s}.$$

We conclude from Fourier inversion formula that $u$ is bounded an continuous (by dominated convergence) and tends to zero as $|x| \to +\infty$ (Riemann-Lebesgue which is trivial for $u \in \mathcal{D}(\mathbb{R}^d)$ by integration by parts, and then follows by density).

case $0 < s < \frac{d}{2}$. Lemma 4.2.1 ensures that $H^s \subset L^{p_c}$ continuously. But by definition $H^s \subset L^2$ continuously, and hence by Hölder, $H^s \subset L^p$ for all $p \in [2, p_c]$.

case $s > \frac{d}{2}$. Let $2 \leq p < +\infty$ then $\sigma = \frac{d}{2} - \frac{d}{p}$ satisfies $0 \leq \sigma < \frac{d}{2} = s$ and $p_c(\sigma) = p$ and hence by the previous step $H^s(\mathbb{R}^d) \subset H^\sigma(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ continuously. ☐

We now turn to the heart of the matter which is the proof of Lemma 4.2.1. The difficulty is of course to understand how the Fourier transform can yield control in an $L^p$ space when $2 < p < +\infty$. The proof we present relies on the real interpolation method which goes beyond the scope of these series of lectures but which is an illustration of the strength of Fourier base localization techniques and the splitting in high and low frequencies for the analysis of functions.

*Proof of Lemma 4.2.1.* Assume without loss of generality that $\|f\|_{\dot{H}^s} = 1$. Pick $A > 0$ and split the function $f$ in low and high frequencies:

$$f = f_{1,A} + f_{2,A} \quad \text{with} \quad f_{1,A} = \mathcal{F}^{-1}(\mathbf{1}_{B(0,A)}\widehat{f}) \quad \text{et} \quad f_{2,A} = \mathcal{F}^{-1}(\mathbf{1}_{{}^c B(0,A)}\widehat{f}). \tag{4.10}$$

Since the Fourier transform of $f_{1,A}$ has compact support, the function $f_{1,A}$ is bounded and more precisely:

$$\begin{aligned} \|f_{1,A}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{1,A}}\|_{L^1} \leq (2\pi)^{-d}\int_{B(0,A)} |\xi|^{-s}|\xi|^s|\widehat{f}(\xi)|\,d\xi \\ &\leq (2\pi)^{-d}\Big(\int_{B(0,A)} |\xi|^{-2s}\,d\xi\Big)^{\frac12} \leq C_s A^{\frac{d}{2}-s}\|f\|_{\dot{H}^s}. \end{aligned} \tag{4.11}$$

The triangle inequality ensures that for all $A > 0$,

$$\{|f| > \lambda\} \subset \{|f_{1,A}| > \lambda/2\} \cup \{|f_{2,A}| > \lambda/2\},$$

and hence (4.11) ensures:

$$A \leq A_\lambda \overset{\text{def}}{=} \left(\frac{\lambda}{2C_s}\right)^{\frac{p}{d}} \Longrightarrow \left|\left\{|f_{1,A}| > \frac{\lambda}{2}\right\}\right| = 0.$$

We conclude by (1.16):

$$\|f\|_{L^p}^p \leq p\int_0^\infty \lambda^{p-1}\left|\left\{|f_{2,A_\lambda}| > \frac{\lambda}{2}\right\}\right|\,d\lambda.$$

High frequencies are now controlled using the Bienaymé-Tchebychev inequality in $L^2$:

$$\left|\left\{|f_{2,A_\lambda}| > \frac{\lambda}{2}\right\}\right| = \int_{\{|f_{2,A_\lambda}|>\frac{\lambda}{2}\}} 1\,dx \leq \int_{\{|f_{2,A_\lambda}|>\frac{\lambda}{2}\}} \frac{4|f_{2,A_\lambda}(x)|^2}{\lambda^2}\,dx \leq 4\frac{\|f_{2,A_\lambda}\|_{L^2}^2}{\lambda^2}.$$

Hence

$$\|f\|_{L^p}^p \leq 4p\int_0^\infty \lambda^{p-3}\|f_{2,A_\lambda}\|_{L^2}^2\,d\lambda. \tag{4.12}$$

We recall Plancherel

$$(2\pi)^d\|f_{2,A_\lambda}\|_{L^2}^2 = \int_{\{|\xi|\geq A_\lambda\}} |\widehat{f}(\xi)|^2\,d\xi$$

which injected into (4.12) yields

$$(2\pi)^d\|f\|_{L^p}^p \leq 4p\int_{\mathbb{R}_+\times\mathbb{R}^d} \lambda^{p-3}\mathbf{1}_{\{(\lambda,\xi)\,/\ |\xi|\geq A_\lambda\}}(\lambda,\xi)|\widehat{f}(\xi)|^2\,d\xi\,d\lambda.$$

Now by definition of $A_\lambda$

$$|\xi| \geq A_\lambda \Longleftrightarrow \lambda \leq C_\xi \overset{\text{def}}{=} 2C_s|\xi|^{\frac{d}{p}}.$$

and hence Fubini ensures

$$(2\pi)^d \|f\|_{L^p}^p \leq 4p \int_{\mathbb{R}^d} \left( \int_0^{C_\xi} \lambda^{p-3}\, d\lambda \right) |\widehat{f}(\xi)|^2 \, d\xi \leq \frac{4p}{p-2} (2C_s)^{p-2} \int_{\mathbb{R}^d} |\xi|^{\frac{d(p-2)}{p}} |\widehat{f}(\xi)|^2 \, d\xi.$$

Since $2s = \dfrac{d(p-2)}{p}$, the estimate (4.9) is proved. □

### 4.2.2 Corollaries of Sobolev injections

We give two elementary but very useful corollaries of the Sobolev injection Theorem.

**Theoreme 4.2.2** (Dual Sobolev injection). *Let $p \in ]1,2]$, then $L^p(\mathbb{R}^d)$ embeds continuously into $H^{-s}(\mathbb{R}^d)$ with $s = d/p - d/2$.*

*Proof of Theorem 4.2.2.* By density, we need only show that there exists $C > 0$ such that

$$\forall u \in \mathcal{S}, \quad \|u\|_{H^{-s}} \leq C\|u\|_{L^p}. \tag{4.13}$$

By Proposition 4.1.4,

$$\|u\|_{H^{-s}} = (2\pi)^d \sup_{\substack{\varphi \in \mathcal{S} \\ \|\varphi\|_{H^s} \leq 1}} \int u\, \varphi \, dx$$

and hence by Hölder,

$$\|u\|_{H^{-s}} \leq (2\pi)^d \sup_{\substack{\varphi \in \mathcal{S} \\ \|\varphi\|_{H^s} \leq 1}} \|u\|_{L^p} \|\varphi\|_{L^{p'}}.$$

Since $d/p' = d/2 - s$, the above Sobolev injections ensure that there exists $C > 0$ such that

$$\forall \phi \in \mathcal{S}, \quad \|\varphi\|_{L^{p'}} \leq C\|\varphi\|_{H^s},$$

and the claim is proved. □

*Remark.* Since Lemma 4.2.1 involves only the homogeneous Sobolev norms, we may replace $\|u\|_{H^{-s}}$ by $\|u\|_{\dot{H}^{-s}}$ in (4.13). This allows us to recover some of the exponents of the Hardy-Littlewood-Sobolev inequality, see exercice 4.20.

A second corollary are the celebrated Gagliardo-Nirenberg interpolation inequalities which are everywhere in the study of non linear problems.

**Corollary 4.2.1** (Gagliardo-Nirenberg interpolation estimate). *Let*

$$2^* = \begin{cases} +\infty & \text{for } d = 1, 2, \\ \frac{2d}{d-2} & \text{for } d \geq 3. \end{cases}$$

*If $2 \leq p < 2^*$, then*

$$\forall u \in H^1(\mathbb{R}^d), \quad \|u\|_{L^p} \leq C\|u\|_{L^2}^{1-\sigma} \|\nabla u\|_{L^2}^{\sigma} \quad \text{with} \quad \sigma = \frac{d(p-2)}{2p} \cdot \tag{4.14}$$

*Remark.* $2^*$ is the universal notation for the homogeneous Sobolev injection $\dot{H}^1(\mathbb{R}^d) \subset L^{2^*}(\mathbb{R}^d)$ in dimension $d \geq 3$. See Lemma 4.4.1 for the generalization to the non Hilbertian setting.

*Proof of 4.2.1.* The estimate (4.9) yields

$$\|u\|_{L^p} \leq C\|u\|_{\dot{H}^\sigma}.$$

Since $2 \leq p < 2^*$ ensures $\sigma \in [0, 1[$ and arguing like for the proof of (4.5) with $|\xi|$ instead of $\langle \xi \rangle$, we obtain

$$\|u\|_{\dot{H}^\sigma} \leq \|u\|_{L^2}^{1-\sigma} \|u\|_{\dot{H}^1}^{\sigma},$$

and the claim follows. □

## 4.3 Compactness of Sobolev embeddings

We are now in position to collect the fruits of the previous chapters and prove the local in space compactness of the Sobolev injection which is the celebrated Rellich Theorem. This is an absolutely fundamental tool for the study of linear and non linear physical models.

### 4.3.1 Ascoli-Arzela Theorem

Let us start with recalling a classical compactness result for sequences of continuous functions: the Ascoli-Arzela Theorem.

**Theoreme 4.3.1** (Ascoli-Arzela theorem). *Let $d, p \geq 1$ and $\overline{B}_R = \{x \in \mathbb{R}^d, \quad \|x\| \leq R\}$. Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of continuous maps from $B_R$ into $\mathbb{R}^p$ i.e.*

$$\sup_{n\geq 1} \|f_n\|_{L^\infty(\overline{B}_R)} < +\infty.$$

*Assume that $(f_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous :*

$$\forall \varepsilon > 0, \quad \exists \eta > 0 \text{ such that } \forall n \in \mathbb{N}, \quad (\|x - y\| < \eta \Longrightarrow \|f_n(x) - f_n(y)\| < \varepsilon). \tag{4.15}$$

*Then there exists $f \in \mathcal{C}(\overline{B}_R; \mathbb{R}^p)$ and a subsequence $(f_{\phi(n)})_{n\in\mathbb{N}}$ such that*

$$f_{\phi(n)} \to f \quad \text{uniformly on } \overline{B}_R.$$

*Remark.* In other words, the subset of uniformly equicontinuous functions in the unit ball of the Banach space $\mathcal{C}(\overline{B}_R; \mathbb{R}^p)$ is relatively compact in $\mathcal{C}(\overline{B}_R; \mathbb{R}^p)$. It is easily seem that (4.15) is in fact necessary and sufficient. The equicontinuity assumption (4.15) should be thought of as a weak form of a derivative bound: if $\sup_{n\in\mathbb{N}} \|\nabla f_n\|_{L^\infty(\overline{B}_R)} < +\infty$, then (4.15) holds. The typical obstruction to the convergence of subsequences is the presence of high oscillations $f_n(x) = \sin(nx)$ (see exo 2.4).

*Proof of 4.3.1.* This is a diagonal extraction argument. Let $m \geq 1$ and $\varepsilon_m = \frac{1}{m}\cdot$ The ball $\overline{B}_R$ is compact, hence we can extract from $\overline{B}_R \subset \bigcup_{x\in\overline{B}} B(x, \varepsilon_m)$ a finite covering. Let $(x_i^{(m)})_{1\leq i\leq N(m)}$ such that

$$\overline{B}_R \subset \bigcup_{i=1}^{N(m)} B(x_i^{(m)}, \varepsilon_m).$$

Let $m = 1$, then the $N(1)$ sequences $\left(f_n(x_i^{(1)})\right)_{n\geq 1}$, $1 \leq i \leq N(1)$ are bounded in $\mathbb{R}^p$, and hence there exists an extraction $\phi_1(n)$ such that

$$\forall 1 \leq i \leq N(1), \quad f_{\phi_1(n)}(x_i^{(1)}) \longrightarrow f_{i,\infty}^{(1)} \quad \text{as} \quad n \longrightarrow +\infty.$$

By induction on $m$, we construct subsequences $\phi_1, \cdots, \phi_m$ such that:

$$\forall 1 \leq i \leq N(m), \quad f_{\phi_1\circ\cdots\circ\phi_m(n)}(x_i^{(m)}) \to f_{i,\infty}^{(m)} \quad \text{when} \quad n \to +\infty.$$

The diagonal map $\phi(n) \stackrel{\text{def}}{=} \phi_1 \circ \ldots \phi_n(n)$ satisfies by construction

$$\forall 1 \leq m, \quad \forall 1 \leq i \leq N(m), \quad f_{\phi(n)}(x_i^{(m)}) \longrightarrow f_{i,\infty}^{(m)} \quad \text{as} \quad n \to +\infty. \tag{4.16}$$

We now claim that $(f_{\phi(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $(\mathcal{C}(\overline{B}_R;\mathbb{R}^p),\|\cdot\|_{L^\infty})$ which concludes the proof. Indeed, let $\varepsilon>0$ and $\eta=\eta(\varepsilon)$ given by (4.15). Let $m=m(\varepsilon)$ such that $\varepsilon_m<\eta$. Let $x\in\overline{B}_R$, then $\exists i\in[1,N(m)]$ such that $\|x-x_i^{(m)}\|<\eta$ and hence by (4.15), for all $n\geq 1$,

$$\begin{aligned}
& \|f_{\phi(n)}(x)-f_{\phi(p)}(x)\| \\
\leq\ & \|f_{\phi(n)}(x)-f_{\phi(n)}(x_i^{(m)})\|+\|f_{\phi(n)}(x_i^{(m)})-f_{\phi(p)}(x_i^{(m)})\|+\|f_{\phi(p)}(x_i^{(m)})-f_{\phi(p)}(x)\| \\
\leq\ & 2\varepsilon+\|f_{\phi(n)}(x_i^{(m)})-f_{\phi(p)}(x_i^{(m)})\|.
\end{aligned}$$

But the sequence $(f_{\phi(n)}(x_i^{(m)}))_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}^p$ and hence a Cauchy sequence, and hence for all $n,p\geq P(\varepsilon)$ large enough:

$$\forall 1\leq i\leq N(m),\quad \|f_{\phi(n)}(x_i^{(m)})-f_{\phi(p)}(x_i^{(m)})\|\leq\varepsilon.$$

We conclude

$$\forall n,p\geq P(\varepsilon),\quad \forall x\in\overline{B}_R,\quad \|f_{\phi(n)}(x)-f_{\phi(p)}(x)\|\leq 3\varepsilon,$$

and the claim is proved. □

### 4.3.2 Compactness of the convolution

The convolution operation is the canonical compact operator since it naturally gains derivtiaves. We give below one compactness result which follows from Ascoli, we refer to exo 2.10.

**Proposition 4.3.1** (Compactness of the convolution)**.** *Soit $d\geq 1$, $1\leq p\leq+\infty$ et $\overline{B}_R=\{x\in\mathbb{R}^d,\ \ \|x\|\leq R\}$. Let $\psi\in\mathcal{S}(\mathbb{R}^d)$ and $T_\psi(f)=\psi\star f$, then*

$$T_\psi:(L^p(\mathbb{R}^d),\|\cdot\|_{L^p(\mathbb{R}^d)})\to(\mathcal{C}(\overline{B}_R;\mathbb{R}),\|\cdot\|_{L^\infty(\mathbb{R}^d)})$$

*is compact. In other words, let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mathbb{R}^d)$, then we can extract a subsequence $(f_{\phi(n)})_{n\in\mathbb{N}}$ such that $(\psi\star f_{\phi(n)})_{n\in\mathbb{N}}$ converge uniformly in $\overline{B}_R$.*

*Proof of Proposition 4.3.1.* Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(\mathbb{R}^d)$. Let us show that $\psi\star f_n\in\mathcal{C}^1(\mathbb{R}^d)$ and satisfies the assumptions of Theorem 4.3.1. Fix $n\in\mathbb{N}$ and assume $p$ finite. Then $\mathcal{D}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ and hence there exist $f_{n,\eta}\in\mathcal{D}(\mathbb{R}^d)$ with $f_{n,\eta}\to f_n$ in $L^p(\mathbb{R}^d)$ as $\eta$ to 0. Then $\psi\star f_{n,\eta}\in\mathcal{S}(\mathbb{R}^d)$ from Lemma 3.1.2 and by Young

$$\|\psi\star f_n-\psi\star f_{n,\eta}\|_{L^\infty(\mathbb{R}^d)}\leq\|f_n-f_{n,\eta}\|_{L^p(\mathbb{R}^d)}\|\psi\|_{L^{p'}(\mathbb{R}^d)}\to 0\ \text{ as }\ \eta\to 0.$$

Hence $\psi\star f_n$ is the uniform limit of a sequence of continuous functions and is therefore continuous. Using

$$\partial_i(\psi\star f_{n,\eta})=\partial_i\psi\star f_{n,\eta}$$

ensures similarly that $\psi\star f_n$ is $\mathcal{C}^1$. If $p=+\infty$, the same holds by deriving directly below the integral. For $1\leq p\leq+\infty$, we moreover obtain the uniform bounds

$$\begin{aligned}
\|\psi\star f_n\|_{L^\infty(\mathbb{R}^d)} &\leq \|\psi\|_{L^{p'}(\mathbb{R}^d)}\|f_n\|_{L^p(\mathbb{R}^d)}\leq C,\\
\|\nabla(\psi\star f_n)\|_{L^\infty(\mathbb{R}^d)} = \|\nabla\psi\star f_n\|_{L^\infty(\mathbb{R}^d)} &\leq \|\nabla\psi\|_{L^{p'}(\mathbb{R}^d)}\|f_n\|_{L^p(\mathbb{R}^d)}\leq C.
\end{aligned}$$

Hence the sequence $(\psi\star f_n)_{n\in\mathbb{N}}$ is uniformly bounded and equicontinous on $\overline{B}_R$, and the conclusion follows from Theorem 4.3.1. □

### 4.3.3 Local compactness of the Sobolev embedding

We now turn to the local compactness in space of Sobolev embeddings: Rellich's Theorem. Locality is in space.

**Definition 4.3.1** ($L^p_{loc}$ convergence)**.** *Let $1 \leq p < +\infty$ and $\Omega$ be an open subset $\mathbb{R}^d$. We say that a sequence $(f_n)_{n\in\mathbb{N}} \in L^p_{\mathrm{loc}}(\Omega)$ converges to $f$ in $L^p_{loc}(\Omega)$ iff*

$$\forall K \ \ compact\ de \ \ \Omega, \quad f_n \to f \ \ in \ \ L^p(K). \tag{4.17}$$

**Theoreme 4.3.2** (Local compactness of the $H^s$ Sobolev injection)**.** *Let $d \geq 1$, $s > 0$ and*

$$p_c = \begin{cases} \frac{2d}{d-2s} & for \ \ s < \frac{d}{2}, \\ +\infty & otherwise. \end{cases}$$

*Then the embedding*

$$H^s(\mathbb{R}^d) \hookrightarrow L^p_{loc}(\mathbb{R}^d) \ \ \text{is compact for all} \ \ 1 \leq p < p_c.$$

*In other words, let $(f_n)_{n\in\mathbb{N}}$ bounded in $H^s(\mathbb{R}^d)$, then there exists $f \in H^s(\mathbb{R}^d)$ and a subsequence $(f_{\phi(n)})_{n\in\mathbb{N}}$ such that:*

$$\begin{array}{lll} f_{\varphi(n)} \rightharpoonup f & dans & H^s(\mathbb{R}^d), \\ f_{\varphi(n)} \to f & dans & L^p_{loc}(\mathbb{R}^d) \ \ \forall 1 \leq p < p_c. \end{array}$$

*If $s > \frac{d}{2}$ then the convergence is unifom on any compact set of $\mathbb{R}^d$.*

*Proof of Theorem 4.3.2.* The key to the proof is the fact that the map

$$\mathrm{Id} : (H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)}) \to L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)}) \tag{4.18}$$

with $\overline{B}_R \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d,\ \|x\| \leq R\}$ is the uniform limit of convolution operators satisfiying the assumptions of Proposition 4.3.1.

**step 1** Compactness. Fix $\zeta \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ positive with

$$\zeta(x) = \begin{cases} 1 & \text{pour} & \|x\| \leq 1, \\ 0 & \text{pour} & \|x\| \geq 2 \end{cases} \quad \text{et} \quad \int_{\mathbb{R}^d} \zeta(x)dx = 1, \tag{4.19}$$

and let the regularizing sequence:

$$\zeta_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0. \tag{4.20}$$

Let

$$T_\varepsilon(f) = \zeta_\varepsilon \star f,$$

then from Theorem 3.1.2

$$\forall f \in L^2(\mathbb{R}^d), \ \ T_\varepsilon(f) \to f \ \ \text{dans} \ \ L^2(\mathbb{R}^d) \ \ \text{quand} \ \ \varepsilon \to 0.$$

Let $s < \frac{d}{2}$. We claim the uniform statement:

$$\sup_{\|f\|_{H^s}\leq 1} \|T_\varepsilon f - f\|_{L^2(\mathbb{R}^d)} \to 0 \ \ \text{quand} \ \ \varepsilon \to 0 \ \ \text{pour} \ \ 0 < s \leq \frac{d}{2}\cdot \tag{4.21}$$

Assume (4.21) which is proved below, then $\forall R > 0$, the map defined in (4.18) is the uniform limit of the sequence $T_\varepsilon$. By Proposition 4.3.1, foall $\varepsilon > 0$ , $T_\varepsilon$ is *compact* from $(L^2(\mathbb{R}^d), \|\cdot\|_{L^2(\mathbb{R}^d)})$ into $\mathcal{C}(\overline{B}_R, \|\cdot\|_{L^\infty(\overline{B}_R)})$ and hence a fortiori from $(H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)})$ into $L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)})$. Hence

$$\mathrm{Id} : (H^s(\mathbb{R}^d), \|\cdot\|_{H^s(\mathbb{R}^d)}) \to L^2(\overline{B}_R, \|\cdot\|_{L^2(\overline{B}_R)}) \text{ est compacte} \tag{4.22}$$

as the uniform limit of compact operator in view of Proposition 2.1.1.
*Proof of* (4.21). We compute

$$\widehat{\zeta_\varepsilon}(\xi) = \widehat{\zeta}(\varepsilon\xi) \ \text{ et } \ (\widehat{T_\varepsilon f} - \widehat{f})(\xi) = (1 - \widehat{\zeta}(\varepsilon\xi))\widehat{f}(\xi)$$

and hence by Plancherel:

$$\begin{aligned}(2\pi)^d \int_{\mathbb{R}^d} |T_\varepsilon(f) - f|^2 dx &= \int_{\mathbb{R}^d} |\widehat{T_\varepsilon f} - \widehat{f}|^2 \, d\xi = \int_{\mathbb{R}^d} |1 - \widehat{\zeta}(\varepsilon\xi)|^2 |\widehat{f}(\xi)|^2 \, d\xi \\ &\leq \|f\|^2_{H^s(\mathbb{R}^d)} \sup_{\xi\in\mathbb{R}^d} \left[\frac{|1 - \hat{\zeta}(\varepsilon\xi)|^2}{\langle\xi\rangle^{2s}}\right].\end{aligned} \tag{4.23}$$

Since $\zeta \in \mathcal{S}$ implies $\widehat{\zeta} \in \mathcal{S}$ and

$$\widehat{\zeta}(0) = \int_{\mathbb{R}^d} \zeta(x) \, dx = 1 \tag{4.24}$$

we easily conclude [3] using $s > 0$,

$$\sup_{\xi\in\mathbb{R}^d} \left[\frac{|1 - \widehat{\zeta}(\varepsilon\xi)|^2}{\langle\xi\rangle^{2s}}\right] \to 0 \ \text{ when } \ \varepsilon \to 0,$$

and (4.21) is proved.

**step 2** Strong convergence in $L^p_{\mathrm{loc}}$. Let now $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $H^s$. By weak compactness of the unit ball of $H^s(\mathbb{R}^d)$, there exists $f \in H^s(\mathbb{R}^d)$ and an extraction $\psi(n)$ such that

$$f_{\psi(n)} \rightharpoonup f \ \text{ in } \ H^s(\mathbb{R}^d). \tag{4.25}$$

Letting $R = m$, we construct by induction on $m \geq 1$ using (4.22) extractions $\phi_1, \cdots, \phi_m, \cdots$, such that

$$\forall m \geq 1, \quad f_{\psi\circ\phi_1\circ\cdots\circ\phi_m(n)} \to f \ \text{ in } \ L^2(\overline{B}_m) \ \text{ as } \ m \to +\infty.$$

The fact that the local strong limit is necessarily given by $f$ follows from the uniqueness of the limit in the sense of distributions. The sequence $(f_{\phi(n)})_{n\in\mathbb{N}}$ where

$$\phi(n) = \psi \circ \phi_1 \circ \cdots \circ \phi_n(n)$$

hence satisfies by construction

$$f_{\phi(n)} \rightharpoonup f \ \text{ dans } \ H^s(\mathbb{R}^d), \quad f_{\phi(n)} \to f \ \text{ in } \ L^2_{loc}(\mathbb{R}^d), \tag{4.26}$$

and hence using Hölder on a given compact of $\mathbb{R}^d$:

$$f_{\phi(n)} \to f \ \text{ in } \ L^p_{loc}(\mathbb{R}^d) \ \text{ pour } \ 1 \leq p \leq 2.$$

[3] cut in $|\xi| \leq \frac{1}{\sqrt{\varepsilon}}$ and $|\xi| \geq \frac{1}{\sqrt{\varepsilon}}$.

Let $2 \leq p < p_c$ and $0 < \alpha \leq 1$ with

$$\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{p_c},$$

let $K$ be a compact of $\mathbb{R}^d$. Hölder and Sobolev (4.8) and the convergence (4.26) yield:

$$\begin{aligned}
\|f_{\phi(n)} - f\|_{L^p(K)} &\leq \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)} \|f_{\phi(n)} - f\|^{1-\alpha}_{L^{p_c}(K)} \\
&\leq C_p \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)} (\|f_{\phi(n)}\|^{1-\alpha}_{H^s(\mathbb{R}^d)} + \|f\|^{1-\alpha}_{H^s(\mathbb{R}^d)}) \\
&\leq 2C_p (\sup_n \|f_n\|_{H^s})^{1-\alpha} \|f_{\phi(n)} - f\|^{\alpha}_{L^2(K)} \to 0 \text{ quand } n \to +\infty.
\end{aligned}$$

The same proof applies for $s > \frac{d}{2}$, once proved that

$$\sup_{\|f\|_{H^s} \leq 1} \|T_\varepsilon f - f\|_{L^\infty(\mathbb{R}^d)} \to 0 \text{ when } \varepsilon \to 0 \text{ for } s > \frac{d}{2}\cdot \tag{4.27}$$

□

Theorem 4.3.2 is essentially optimal in the following sense.

*(i) Sobolev in never compact on* $L^p(\mathbb{R}^d)$. The continuous injection $H^s(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is never compact due to the action of the translation invariance group. Let

$$f_n(x) = f(x - x_n), \quad |x_n| \to +\infty$$

for a given non zero profile $f$, then $f_n \rightharpoonup 0$ in $H^s$ but $\forall p \geq 1$, $\|f_n\|_{L^p} = \|f\|_{L^p}$ and hence the sequence does not strongly converge in $L^p$. This type of default of compactness can be avoided using symmetry assumptions, see Proposition 7.1.1. The description of the default of compactness of the Sobolev injection $H^1(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$ is the heart of concentration-compactness techniques, see chapter 9.

*(ii) Critical Sobolev is never compact.* The critical continuous embedding $H^s(\mathbb{R}^d) \subset L^{p_c}_{loc}(\mathbb{R}^d)$ is never compact due to the action of the group of dilations. Indeed, let $f \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ non zero with support in $\overline{B}_1$, let $(\lambda_n)_{n\in\mathbb{N}} \to 0$ as $n \to +\infty$ and consider

$$f_n(x) \stackrel{\text{def}}{=} \lambda_n^{s-\frac{d}{2}} f\left(\frac{x}{\lambda_n}\right).$$

Then $\text{Supp} f_n \subset \overline{B}_1$ and is bounded from direct check in $H^s$, but

$$|f_n|^{p_c} = \frac{1}{\lambda_n^d} |f|^{p_c}\left(\frac{x}{\lambda_n}\right) \rightharpoonup \|f\|^{p_c}_{L^{p_c}(\mathbb{R}^d)} \delta_0 \text{ in } \mathcal{D}'(\mathbb{R}^d)$$

and hence does not admit any strongly converging subsequence in $L^{p_c}(B_1)$.

### 4.3.4 The case of a bounded domain

Simple consequences of the previous section apply to study Sobolev spaces on a bounded domain as well.

**Definition 4.3.2** ($H_0^1(\Omega), H^{-1}(\Omega)$). *Let $\Omega$ be an open subset of $\mathbb{R}^d$. We define $H_0^1(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ for the $H^1(\mathbb{R}^d)$ norm The space $H^{-1}(\Omega)$ is its topological dual, or equivalently the space of distributions $u \in \mathcal{D}'(\Omega)$ such that*

$$\|u\|_{H^{-1}(\Omega)} \stackrel{def}{=} \sup_{\substack{\varphi\in\mathcal{D}(\Omega)\\ \|\varphi\|_{H^1(\mathbb{R}^d)}\leq 1}} |\langle u,\varphi\rangle| < \infty.$$

Recall that Proposition 4.1.3 ensures that $H^1(\mathbb{R}^d)$ is the closure of $\mathcal{D}(\mathbb{R}^d)$ for the $H^1(\mathbb{R}^d)$ norm, and hence $H_0^1(\Omega)$ can be identified as a closed vectorial subset of $H^1(\mathbb{R}^d)$. Hence the decomposition

$$H^1(\mathbb{R}^d) = H_0^1(\Omega) \oplus (H_0^1(\Omega))^\perp.$$

We conclude that $H_0^1(\Omega)$ equipped with the scalar product

$$(u,v) \longmapsto \int_\Omega u\,\overline{v}\,dx + \sum_{1\leq j\leq d}\int_\Omega \partial_j u\,\overline{\partial_j v}\,dx$$

is a Hilbert space. If $\Omega$ is bounded, the injection $H_0^1(\Omega) \subset H^1(\mathbb{R}^d)$ being continuous, Theorem 4.3.2 with Remark 4.2.2 yields:

**Theorem** (Kato-Rellich). *Let $\Omega$ be a bounded open set of $\mathbb{R}^d$.*

*(i) For $d=1$, the injection $H_0^1(\Omega) \hookrightarrow \mathcal{C}(\Omega)$ is compact.*

*(ii) For $d=2$ and $2 \leq p < +\infty$, the injection $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*

*(iii) For $d \geq 3$, let*

$$2^* \stackrel{def}{=} \frac{2d}{d-2} \in ]2,+\infty[$$

*be the critical exponent. Then for all $p \in [1, 2^*]$, the space $H_0^1(\Omega)$ embeds continuously into $L^p(\Omega)$ with compact embedding if $p < 2^*$.*

*Remark.* One can show that more generally, the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ (or $L^p(\Omega)$ with $p < 2^*$) is compact as soon as $\Omega$ has finite measure (exercice 4.12).

*Remark.* By duality, we also obtain that the embedding of $L^p(\Omega)$ into $H^{-1}(\Omega)$ is compact if $p > \frac{2d}{2+d}$.

The following fundamental result ensures that the homogeneous Sobolev norm is a norm on $H_0^1(\Omega)$ which is the weak form of the "zero boundary" condition on the frontier of $\Omega$.

**Theoreme 4.3.3** (Poincaré inequality). *Let $\Omega$ be a bounded open set of $\mathbb{R}^d$. Then there exists $\lambda_1(\Omega) > 0$ such that:*

$$\forall u \in H_0^1(\Omega)\,,\ \|\nabla u\|^2_{L^2(\Omega)} \geq \lambda_1(\Omega)\|u\|^2_{L^2(\Omega)} \tag{4.28}$$

*where*

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{j=1}^d \|\partial_j u\|^2_{L^2(\Omega)}.$$

*Remark.* $\lambda_1(\Omega) > 0$ is the first eigenvalue of the Laplace operator with Dirichlet (ie zero) boundary condition. The estimate (4.28) is a first example of spectral gap estimate [4]. The dependance of $\lambda_1(\Omega)$ on the domain $\Omega$ is a subtle geometric problem (shape optimization) which is not completely understood.

[4] We refer to G. Allaire [1], Chap. 7, for more examples.

*Proof of Theorem 4.3.3.* We give two proofs: a quantitative one, and a qualitative one (see exercice 4.12 for more applications) which is a first intrusion into variational methods.
*Quantitative proof.* Since $\Omega$ is bounded, $\Omega \subset ]0, R[\times\mathbb{R}^{d-1}$ for some $R$ large enough. Let $u \in \mathcal{D}(\subset ]0, R[\times\mathbb{R}^{d-1})$, then

$$u(x_1, \cdots, x_d) = \int_0^{x_1} \frac{\partial u}{\partial y_1}(y_1, x_2, \cdots, x_d)\, dy_1.$$

and hence from Cauchy Schwarz:

$$|u(x_1, \cdots, x_d)|^2 \leq R \int_0^R \left| \frac{\partial u}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.$$

Since Supp $u \subset ]0, R[\times\mathbb{R}^{d-1}$, integration in $x_1$ yields:

$$\int_{\mathbb{R}} |u(x_1, \cdots, x_d)|^2 dx_1 \leq R^2 \int_0^R \left| \frac{\partial u}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1.$$

and then integrating with respect to the $d-1$ remaining variables,

$$\begin{aligned} \int_{\mathbb{R}^d} |u(x_1, \cdots, x_d)|^2 dx &\leq R^2 \int_{\Omega} \left| \frac{\partial u}{\partial y_1}(y_1, x_2, \cdots, x_d) \right|^2 dy_1\, dx_2 \cdots dx_d \\ &\leq R^2 \|\partial_1 u\|^2_{L^2(\Omega)}. \end{aligned}$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, (4.28) is proved.
*Qualitative proof.* Let

$$\lambda_1(\Omega) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Assume by contradiction that $\lambda_1(\Omega) = 0$. Then there exists a sequence $(u_n)_{n\geq 1}$ with

$$\forall n \in \mathbb{N}^*,\ \|u_n\|_{L^2(\Omega)} = 1 \text{ and } \|\nabla u_n\|_{L^2(\Omega)} \leq \frac{1}{n}.$$

Since $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, we extract from Kato-Rellich $(u_{\phi(n)})_{n\in\mathbb{N}}$ and $u \in H_0^1(\Omega)$ with

$$u_{\phi(n)} \rightharpoonup u \text{ in } H_0^1(\Omega) \qquad \text{and} \qquad u_{\phi(n)} \to u \text{ in } L^2(\Omega).$$

The weak convergence in $H_0^1(\Omega)$ implies convergence in $\mathcal{D}'(\Omega)$, and hence $\nabla u_{\phi(n)} \rightharpoonup \nabla u$ in $L^2(\Omega)$ from which by lower semi continuity of the $L^2$ norm when passing to the weak limit:

$$\int_{\Omega} |\nabla u|^2\, dx \leq \liminf_{n\to+\infty} \int_{\Omega} |\nabla u_n|^2\, dx = 0.$$

Hence $u \in H_0^1(\Omega)$ is a constant and this implies $u \equiv 0$. On the other hand, by strong $L^2(\Omega)$ convergence:

$$\int_{\Omega} |u|^2\, dx = \lim_{n\to+\infty} \int_{\Omega} |u_{\phi(n)}|^2\, dx = 1,$$

and a contradiction follows. □

The following corollay follows directly from (4.28).

**Corollary 4.3.1.** *Let $\Omega$ be a bounded subset of $\mathbb{R}^d$, then*

$$(u, v) \longmapsto \sum_{1\leq j\leq d} \int_{\Omega} \partial_j u\, \overline{\partial_j v}\, dx$$

*is a scalar product on $H_0^1(\Omega)$ which defines a norm equivalent to the one of Definition 4.3.2.*

## 4.4 The Sobolev space $W^{k,p}(\mathbb{R}^d)$

Sobolev spaces $H^s(\mathbb{R}^d)$ are built on $L^2(\mathbb{R}^d)$ which makes the Fourier transform techniques particularily useful. For appication to nonlinear problems, the more general $W^{k,p}(\mathbb{R}^d)$ scale based on $L^p(\mathbb{R}^d)$ with $1 \leq p < +\infty$ is useful. To simplify the exposition, we restrict to integer derivatives $k \in \mathbb{N}$. Let us stress that the general theory is the one of Besov spaces which study relies on the Litllewood-Paley decomposition of functions, see for example [2, 31]. One important feature of the exposition below is to provide another proof of the Sobolev injection Theorem based solely on integration by parts due to L. Nirenberg.

### 4.4.1 Definition and Banach space structure

**Definition** (Sobolev space $W^{k,p}(\mathbb{R}^d)$)**.** Let$1 \leq p < +\infty$ and $k \in \mathbb{N}^*$. We let $W^{k,p}(\mathbb{R}^d)$ be the set of functions $f \in L^p(\mathbb{R}^d)$ such that

$$\forall \alpha \in \mathbb{N}^d \text{ with } |\alpha| \leq k, \quad \partial^\alpha f \in L^p(\mathbb{R}^d).$$

**Theoreme 4.4.1.** *Let $1 \leq p < +\infty$ and $k \in \mathbb{N}^*$. Then $(W^{k,p}(\mathbb{R}^d), \|\cdot\|_{W^{k,p}(\mathbb{R}^d)})$ is a Banach space for the norm:*

$$\|f\|_{W^{k,p}(\mathbb{R}^d)} = \left( \sum_{|\alpha| \leq k} \|\partial^\alpha f\|^p_{L^p(\mathbb{R}^d)} \right)^{\frac{1}{p}}.$$

*Moreover, $\mathcal{D}(\mathbb{R}^d)$ is dense in $W^{k,p}(\mathbb{R}^d)$.*

*Proof of Theorem 4.4.1.* The normed vectorial space structure follows Minkowski's inequality. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{k,p}(\mathbb{R}^d)$. Then for all $|\alpha| \leq k$, $(\partial^\alpha f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^d)$, hence it converges to some $g^\alpha$. Let $g$ be the limit of $(f_n)_{n\in\mathbb{N}}$ in $L^p$, then $f_n \rightharpoonup g$ in $\mathcal{D}'(\mathbb{R}^d)$ and hence $\partial^\alpha f_n \rightharpoonup \partial^\alpha g$ in $\mathcal{D}'(\mathbb{R}^d)$ which by uniqueness of the limit in the sense of distributions forces $g^\alpha = \partial^\alpha f$. Hence $\partial^\alpha (f_n)_{n\in\mathbb{N}}$ converges to $g$ in $W^{k,p}(\mathbb{R}^d)$. The density of $\mathcal{D}(\mathbb{R}^d)$ in $W^{k,p}$ follows from Theorem 3.1.2. □

### 4.4.2 Sobolev injections

We now establigh the $L^2$ analogue of Theorem 4.2.1 and we consider for the sake of simplicity integer derivatives only.

**Theoreme 4.4.2** (Injection de Sobolev)**.** *Let $d \geq 1$, $k \in \mathbb{N}^*$ and $1 \leq p < +\infty$.*

*(i) If $p > \frac{d}{k}$ or $p = d = k = 1$ then $W^{k,p}(\mathbb{R}^d)$ embeds continuously into the space $(\mathcal{C}_0(\mathbb{R}^d), \|\cdot\|_{L^\infty(\mathbb{R}^d)})$ of continuous functions on $\mathbb{R}^d$ which tend to $0$ at infinity.*

*(ii) If $1 \leq p < \frac{d}{k}$, let $p_c$ be the critical exponent*

$$-k + \frac{d}{p} = \frac{d}{p_c} \quad \text{i.e. } p_c = \frac{pd}{d - kp} \in ]p, +\infty[.$$

*Then for all $p \leq q \leq p_c$, $W^{k,p}(\mathbb{R}^d)$ embeds continuously into $L^q(\mathbb{R}^d)$.*

*(iii) If $p = \frac{d}{k} \geq 1$ and $d \geq 2$ then for all $p \leq q < +\infty$, $W^{k,p}(\mathbb{R}^d)$ embed continuously into $L^q(\mathbb{R}^d)$.*

Like for Theorem 4.2.1, the heart of the proof is the scale invariant homogeneous Sobolev estimate.

**Lemma 4.4.1** (Homogeneous $W^{1,p}$ injection)**.** *Let $d \geq 1$.*

*(i) If $1 \leq p < d$ and $p^*$ is given by*

$$-1 + \frac{d}{p} = \frac{d}{p^*} \quad i.e. \;\; p^* = \frac{pd}{d-p} \in ]p, +\infty[$$

*then*

$$\forall f \in W^{1,p}(\mathbb{R}^d), \quad \|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C_p \|\nabla f\|_{L^p(\mathbb{R}^d)}. \tag{4.29}$$

*(ii) If $p > d$ and $\alpha = 1 - \frac{d}{p}$, then there holds the uniform Hölder bound:*

$$\forall f \in W^{1,p}(\mathbb{R}^d), \quad \forall (x,y) \in \mathbb{R}^{2d}, \quad |f(x) - f(y)| \leq C_{p,d}|x-y|^\alpha \|\nabla f\|_{L^p(\mathbb{R}^d)} \tag{4.30}$$

*Proof of* (4.4.1). We follow [5][5]. By density it suffices to prove the claim for $f \in \mathcal{D}$.
**step 1** Case $p = 1$ and $d \geq 2$. Let $x \in \mathbb{R}^d$ and denote

$$\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d), \quad 1 \leq i \leq d.$$

Let $(g_1, \ldots, g_d) \in \mathcal{D}(\mathbb{R}^{d-1})$ and $g(x) = \prod_{i=1}^d g_i(\tilde{x}_i)$. Let us show by induction on $d$ that

$$\|g\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^d \|g_i\|_{L^{d-1}(\mathbb{R}^{d-1})}. \tag{4.31}$$

This is straightforward for $d = 2$. We assume $d$ and prove $d+1$. Let us freeze $x_{d+1} \in \mathbb{R}$. Let $x = (x', x_{d+1})$, $x' = (x_1, \cdots, x_d)$, then Hölder with respect to the Lebesgue measure in $\mathbb{R}^d_{x'}$ yields:

$$\int |g(x)|\, dx' \leq \|g_{d+1}\|_{L^d(\mathbb{R}^d)} \left( \int |(g_1 \ldots g_d)(x', x_{d+1})|^{\frac{d}{d-1}} dx' \right)^{\frac{d-1}{d}}.$$

We then apply the induction claim to $|g_1|^{\frac{d}{d-1}}, \ldots, |g_d|^{\frac{d}{d-1}}$ and hence

$$\int |(g_1 \ldots g_d)(x)|^{\frac{d}{d-1}}\, dx' \leq \prod_{i=1}^d \||g_i(\cdot, x_{d+1})|^{\frac{d}{d-1}}\|_{L^{d-1}(\mathbb{R}^{d-1})} = \prod_{i=1}^d \|g_i(\cdot, x_{d+1})|^{\frac{d}{d-1}}_{L^d(\mathbb{R}^{d-1})}.$$

We have obtained for all fixed $x_{d+1}$:

$$\int |g(x', x_{d+1})|\, dx' \leq \|g_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{i=1}^d \|g_i(\cdot, x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}.$$

We now integrate on $x_{d+1}$. Each function $x_{d+1} \mapsto \|g_i(\cdot, x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}$ for $1 \leq i \leq d$ belongs to $L^d(\mathbb{R})$ avec

$$\Big\| \|g_i(\cdot, x_{d+1})\|_{L^d(\mathbb{R}^{d-1})} \Big\|_{L^d(\mathbb{R})} = \|g_i\|_{L^d(\mathbb{R}^d)},$$

[5]another approach is proposed in exercice 4.19.

and hence using Hölder again with $\displaystyle\sum_{i=1}^{d}\frac{1}{d}=1$ yields

$$\int |g(x)|\,dx'\,dx_{d+1} \le \|g_{d+1}\|_{L^d(\mathbb{R}^d)} \int \prod_{i=1}^{d} \|g_i\|_{L^d(\mathbb{R}^{d-1})}\,dx_{d+1} = \prod_{i=1}^{d+1} \|g_i(\cdot,x_{d+1})\|_{L^d(\mathbb{R}^d)}.$$

**step 2** Case $p<d$ and $d\ge 2$. If $p=1$ the estimate (4.29) follows from an integration by parts: for all $1\le i\le d$:

$$\begin{aligned}
|f(x)| &= \left|\int_{-\infty}^{x_i} \partial_i f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_d)\,dt\right| \\
&\le f_i(\tilde{x}_i) \overset{\text{def}}{=} \int_{\mathbb{R}} |\partial_i f(x_1,\dots,x_{i-1},t,x_{i+1},\dots,x_d)|\,dt,
\end{aligned}$$

and hence

$$|f(x)|^d \le \prod_{i=1}^{d} f_i(\tilde{x}_i).$$

We conclude from (4.31):

$$\int |f(x)|^{\frac{d}{d-1}}dx \le \prod_{i=1}^{d} \|f_i\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d-1}} = \prod_{i=1}^{d} \|\partial_i f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}}$$

and hence

$$\|f\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \le \prod_{i=1}^{d} \|\partial_i f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \tag{4.32}$$

and (4.29) is proved for $p=1$ and $d\ge 2$.
If $1<p<d$, we fixe $t>1$ and $f\in\mathcal{D}$ and apply (4.32) to $f|f|^{t-1}$. Using Hölder:

$$\begin{aligned}
\|f\|^t_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} &\le C_{p,t}\prod_{i=1}^{d} \||f|^{t-1}\partial_i f\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \le C_p \prod_{i=1}^{d}\left[\|f\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^d)}\|\partial_i f\|_{L^p(\mathbb{R}^d)}\right]^{\frac{1}{d}} \\
&\le C_{p,t}\|f\|^{t-1}_{L^{p'(t-1)}(\mathbb{R}^d)}\prod_{i=1}^{d}\|\partial_i f\|^{\frac{1}{d}}_{L^p(\mathbb{R}^d)}.
\end{aligned} \tag{4.33}$$

The choice

$$t=\frac{d-1}{d}p^* \ \text{ i.e. } \ \frac{td}{d-1}=p'(t-1)=p^*$$

for which $t\ge 1$ (since $p<d$) ensures:

$$\|f\|_{L^{p^*}(\mathbb{R}^d)} \le C_p\prod_{i=1}^{d}\|\partial_i f\|^{\frac{1}{d}}_{L^p(\mathbb{R}^d)} \le C_p\|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

The estimate (4.29) is thus proved for $f\in\mathcal{D}(\mathbb{R}^d)$, and the general case $f\in W^{1,p}(\mathbb{R}^d)$ follows by density.

**step 3** Case $p > d$. By density we need only treat the case $f \in \mathcal{D}$. The estimate (4.30) is obvious in dimension $d = 1$ since by Hölder:

$$|f(x) - f(y)| = \left| \int_x^y f'(t)dt \right| \le |x-y|^{1/p'} \|f\|_{L^p(\mathbb{R})}.$$

The proof is more subtle in dimension $d \ge 2$. Let $r > 0$ and $Q = [-r, r]^d$, then

$$\forall x \in Q, \quad |f(x) - f(0)| = \left| \int_0^t x \cdot \nabla f(tx)\, dt \right| \le r \int_0^1 |\nabla f(tx)|\, dt. \tag{4.34}$$

Let $\overline{f}_Q$ be the average of $f$ on $Q$

$$\overline{f}_Q = \frac{1}{|Q|} \int_Q f(x)\, dx.$$

Then integrating (4.34) for $x \in Q$ and using Fubini, a change of variables and Hölder:

$$\begin{aligned}
|\overline{f}_Q - f(0)| &\le \frac{r}{|Q|} \int_0^1 \int_Q |\nabla f(tx)|\, dx\, dt = \frac{r}{|Q|} \int_0^1 \frac{1}{t^d} \int_{tQ} |\nabla f(x)|\, dx\, dt \\
&\le \frac{r}{(2r)^d} \int_0^t \|\nabla f\|_{L^p(\mathbb{R}^d)} \frac{|tQ|^{\frac{1}{p'}}}{t^d} dt \le \frac{r^{1-d+\frac{d}{p'}}}{2^{d/p}} \|\nabla f\|_{L^p(\mathbb{R}^d)} \int_0^1 t^{\frac{d}{p'}-d}\, dt \\
&\le C_{p,d}\, r^{1-\frac{d}{p}} \|\nabla f\|_{L^p(\mathbb{R}^d)}
\end{aligned}$$

where we used

$$\frac{d}{p'} - d = -\frac{d}{p} > -1.$$

By translation invariance, we conclude that for all cube $Q$ of size $2r$:

$$\forall x \in Q, \quad |\overline{f}_Q - f(x)| \le C_{p,d}\, r^{1-\frac{d}{p}} \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

and hence

$$\forall (x,y) \in Q^2, \quad |f(x) - f(y)| \le |f(x) - \overline{f}_Q| + |\overline{f}_Q - f(y)| \le C_{p,d}\, r^{1-\frac{d}{p}} \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$

Since two points $(x, y) \in \mathbb{R}^{2d}$ always belong to such a cube with $r = 2|x-y|$, (4.30) is proved. $\square$

*Proof of Theorem 4.4.2.* We detail the proof for $k = 1$. The claim for $k \ge 2$ follows directly by induction on $k$.

(*i*) Case $p > d$. By (4.30), every function $f \in W^{1,p}(\mathbb{R}^d)$ with $p > d$ is Hölderian, and hence continuous. Moreover, if $x \in \mathbb{R}^d$ and $Q = \Pi_{1=1}^d [x_i - 1, x_i + 1]$, then there exists $y \in Q$ such that [6]:

$$|f(y)| \le \frac{1}{|Q|} \int_Q |f(z)| dz \le C_p \|f\|_{L^p(\mathbb{R}^d)}$$

and hence by (4.30):

$$|f(x)| \le |f(y)| + C_{p,d} \|\nabla f\|_{L^p(\mathbb{R}^d)} \le C_{p,d} \|f\|_{W^{1,p}(\mathbb{R}^d)}.$$

This proves that $f$ is bounded and the uniform convergence norm is bounded by the $W^{1,p}$ norm. Since $\mathcal{D}(\mathbb{R}^d)$ is dense in $W^{1,p}$ and the uniform convergence preserves the limit at infinity, we conclude that $f \in \mathcal{C}_0$.

[6] Raisonner par l'absurde.

*(i')* Case $p = d = 1$. For $f \in \mathcal{D}(\mathbb{R}^d)$,

$$|f(x)| \leq \int_{-\infty}^{x} |f'(t)|\, dt,$$

and the claim follows by density

*(ii)* Case $d \geq 2$ and $1 \leq p < d$ with $p \leq p^*$. This follows directly from (4.29) and Hölder.

*(iii)* Cas $p = d \geq 1$. Let $f \in \mathcal{D}(\mathbb{R}^d)$, then by (4.33): $\forall t \geq 1$,

$$\begin{aligned} \|f\|_{L^{\frac{td}{d-1}}(\mathbb{R}^d)} &\leq C_{p,t} \|f\|_{L^{p'(t-1)}(\mathbb{R}^d)}^{\frac{t-1}{t}} \|\nabla f\|_{L^p(\mathbb{R}^d)}^{\frac{1}{t}} \\ &\leq C_{p,t} \left[ \|f\|_{L^{p'(t-1)}(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)} \right] \end{aligned} \tag{4.35}$$

where we used Young $ab \leq \frac{1}{p} a^p + \frac{1}{p'} b^{p'}$. The choice

$$p'(t-1) = p \quad \text{i.e.} \quad t = p = d$$

ensures

$$\|f\|_{L^{\frac{d^2}{d-1}}(\mathbb{R}^d)} \leq C_d \|f\|_{W^{1,p}(\mathbb{R}^d)}.$$

We then iterate the process and apply (4.35) to the sequence $(t_j = d + j)_{j \geq 1}$ which goes to $+\infty$ as $j \to +\infty$.

□

*Remark.* This new injection Theorem allows one to compare the $H^s$ and $W^{1,p}$ ladders, see exercice 4.18.

### 4.4.3 Local compactness of the Sobolev embedding

**Theoreme 4.4.3** (Local compactness of the $W^{1,p}(\mathbb{R}^d)$ injection)**.** *Let $d \geq 1$, $p \geq 1$ et*

$$p^* = \begin{cases} \frac{pd}{d-p} & \text{for } p < d \\ +\infty & \text{otherwise.} \end{cases}$$

*Then for all $1 \leq q < p^*$ the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q_{loc}(\mathbb{R}^d)$ iscompact.*

*Equivalently, for all sequence $(f_n)_{n \in \mathbb{N}}$ bounded in $W^{1,p}(\mathbb{R}^d)$, there exists $f \in W^{1,p}(\mathbb{R}^d)$ and a subsequence $(f_{\phi(n)})_{n \in \mathbb{N}}$ such that:*

$$f_{\varphi(n)} \to f \quad \textit{dans} \quad L^q_{loc}(\mathbb{R}^d), \quad \forall 1 \leq q < p^*. \tag{4.36}$$

*For $p > d$, the convergence is uniform on any compact set of $\mathbb{R}^d$.*

*Proof of Theorem 4.4.3.* Let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence of $W^{1,p}(\mathbb{R}^d)$ and $R > 0$. Let us show that we can extract

$$f_{\varphi(n)} \to f \quad \text{dans} \quad L^q(\overline{B}_R), \quad \forall 1 \leq q < p^*, \tag{4.37}$$

with moreover uniform convergence if $p > d$.
– For $p > d$, the uniform Hölder estimate (4.30) implies that the family $(f_n)_{n\geq 1}$ is equicontinuous on $\overline{B}_R$, and (4.37) follows from Ascoli.
– If $p \leq d$, let $q < p^*$, let $\zeta_\varepsilon$ be a regularizing sequence. Assume

$$\sup_{\|f\|_{W^{1,p}(\mathbb{R}^d)}\leq 1} \|\zeta_\varepsilon \star f - f\|_{L^p(\mathbb{R}^d)} \to 0 \quad \text{quand} \quad \varepsilon \to 0. \tag{4.38}$$

Then Id : $W^{1,p}(\mathbb{R}^d) \to L^p(\overline{B}_R)$ is the uniform limit of the maps $f \mapsto \zeta_\varepsilon \star f$ which by Proposition 4.3.1 are compact from $L^p(\mathbb{R}^d) \to \mathcal{C}(\overline{B}_R)$ and hence a fortiori from $W^{1,p}(\mathbb{R}^d) \to L^p(\overline{B}_R)$. Hence Id : $W^{1,p}(\mathbb{R}^d) \to L^p(\overline{B}_R)$ is compact by Proposition 2.1.1. The convergence (4.36) in $L^p_{loc}(\mathbb{R}^d)$ follows by diagonal extraction on $R_m = m$ like for the proof of Theorem 4.3.2, and then in $L^q_{loc}(\mathbb{R}^d)$ for $1 \leq q < p^*$ by Hölder on a fixed compact set for $1 \leq q \leq p$ and the Sobolev injections for $p \leq q < p^*$. The proof for $p > d$ is similar and left to the reader.
*Proof of* (4.38) *for* $1 \leq p < d$. Changing variable and using (4.19):

$$\begin{aligned} |\zeta_\varepsilon \star f(x) - f(x)|^p &= \left|\int_{\mathbb{R}^d} \zeta(y)(f(x-\varepsilon y) - f(y))\,dy\right|^p \leq \left|\int_{|y|\leq 2} |f(x-\varepsilon y) - f(y)|\,dy\right|^p \\ &\leq C_p \int_{|y|\leq 2} |f(x-\varepsilon y) - f(y)|^p\,dy. \end{aligned} \tag{4.39}$$

Let $h \in \mathbb{R}^d$. For $\varphi \in \mathcal{D}(\mathbb{R}^d)$, we compute:

$$|\varphi(x+h) - \varphi(x)|^p = \left|\int_0^1 h \cdot \nabla\varphi(x+th)\,dt\right|^p \leq |h|^p \int_0^1 |\nabla\varphi(x+th)|^p\,dt$$

and hence changing variable $x \mapsto x + th$,

$$\int_{\mathbb{R}^d} |\varphi(x+h) - \varphi(x)|^p dx \leq \int_0^1 |h|^p \int_{\mathbb{R}^d} |\nabla\varphi(x+th)|^p\,dx\,dt \leq |h|^p \|\nabla\phi\|^p_{L^p(\mathbb{R}^d)}.$$

We conclude

$$\forall f \in W^{1,p}(\mathbb{R}^d), \quad \forall h \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p\,dx \leq |h|^p \|f\|^p_{W^{1,p}(\mathbb{R}^d)}.$$

We inject this estimate into (4.39) and obtain:

$$\int_{\mathbb{R}^d} |\zeta_\varepsilon \star f(x) - f(x)|^p\,dx \quad \leq \quad C_p \int_{\mathbb{R}^d} \left(\int_{|y|\leq 2} |f(x-\varepsilon y) - f(y)|^p\,dy\right) dx \leq C_p |\varepsilon|^p \|f\|^p_{W^{1,p}(\mathbb{R}^d)},$$

and (4.38) is proved. □

### 4.4.4 The case of a bounded domain

Like for $H^1$, the $W^{1,p}$ extends naturally onto a domain.

**Definition 4.4.1.** *Let* $\Omega$ *be an open subset of* $\mathbb{R}^d$ *and* $1 \leq p < \infty$. *The space* $W^{1,p}_0(\Omega)$ *is the closure of* $\mathcal{D}(\Omega)$ *for the norm* $\|\cdot\|_{W^{1,p}(\mathbb{R}^d)}$.

By construction $W^{1,p}_0(\Omega)$ is closed subspace of $W^{1,p}(\mathbb{R}^d)$ for the norm $\|\cdot\|_{W^{1,p}(\mathbb{R}^d)}$ and hence a Banach space. The Sobolev injections of Theorem 4.4.2 yield :

**Theorem** (Kato-Rellich in $W^{1,p}(\Omega)$). *Let $\Omega$ be a bounded set of $\mathbb{R}^d$.*

*(i) For $p > d$, the embedding $W_0^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\Omega)$ is compact.*

*(ii) For $p = d$ and $1 \leq q < +\infty$, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

*(iii) For $1 \leq p < d$, let the critical exponent*

$$p^* = \frac{pd}{d-p},$$

*then for all$q \in [1, p^*]$, $W_0^{1,p}(\Omega)$ embeds continuously intp $L^q(\Omega)$ with compact injection for $1 \leq q < p^*$.*

We can also recover a Poincaré inequality for $p = 2$ which proof of is similar to the one of Theorem 4.3.3

**Theoreme 4.4.4** (Poincaré inequality). *Let $\Omega$ be an open subset of $\mathbb{R}^d$ et $1 \leq p < +\infty$. Then there exists $C(p, \Omega)$ such that*

$$\forall f \in W_0^{1,p}(\Omega), \quad \|f\|_{L^p(\Omega)} \leq C(p, \Omega) \|\nabla f\|_{L^p(\Omega)}.$$

## 4.5 Exercices

**Exercice 4.1.** Show that for all $s \in \mathbb{R}$, $\mathcal{S}$ embeds continuously into $H^s$ $H^s$.

**Exercice 4.2.** Show that for all $s$, the multiplication by $\langle \cdot \rangle^s$ sends continuously $\mathcal{S}$ into itself. Same question with the operator $\langle D \rangle^s$ defined by (4.6). Generalize to $\mathcal{S}'$.

**Exercice 4.3.** We say that a distribution $u \in \mathcal{D}'(\mathbb{R}^d)$ has compact support if there exists $K \subset \mathbb{R}^d$ compact such that $\forall \phi \in \mathcal{D}(\mathbb{R}^d \backslash K)$, $\langle u, \phi \rangle_{\mathcal{D}', \mathcal{D}} = 0$. Show that if $u \in \mathcal{D}'(\mathbb{R}^d)$ has compact support, there exists $s \in \mathbb{R}$ such that $u \in H^s$.

**Exercice 4.4.** Show that the constante 1 does not belong to any $H^s$.

**Exercice 4.5.** Show that the Dirac mass $\delta_0$ belongs to $H^{-\frac{d}{2}-\varepsilon}$ for all $\varepsilon > 0$, nut $\delta_0 \in\notin H^{-\frac{d}{2}}$. Generalize to the derivatives of $\delta_0$.

**Exercice 4.6.** Let $s \leq d/2$, show that $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ is dense in $H^s$.
Hint: study the orthogonal of $\mathcal{D}(\mathbb{R}^d \setminus \{0\})$ and use the preceding exercice.

**Exercice 4.7.** Let $\mathcal{R} = \sum_{i=1}^d x_i \partial_{x_i}$.

$(i)$ Compute $\mathcal{R} |\cdot|^{-2}$.

$(ii)$ Show that for all $f \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx = \int_{\mathbb{R}^d} \frac{f(x) \mathcal{R} f(x)}{|x|^2} \, dx + \frac{d}{2} \int \frac{|f(x)|^2}{|x|^2} \, dx.$$

$(iii)$ For $d \geq 3$, prove the Hardy inequality

$$\forall f \in H^1(\mathbb{R}^d), \quad \left( \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \leq \frac{2}{d-2} \|\nabla f\|_{L^2}. \tag{4.40}$$

$(iv)$ Is it true in dimension $d = 2$?

**Exercice 4.8** (Limit cases of Sobolev injections). $(i)$ Show that $H^{\frac{d}{2}}(\mathbb{R}^d)$ does not embed continuously into $L^\infty(\mathbb{R}^d)$.

$(ii)$ In which Sobolev spaces does $L^1(\mathbb{R}^d)$ embed continuously?

$(iii)$ Let $d \geq 3$, show that $H^1(\mathbb{R}^d)$ does not embed continuously into $L^p(\mathbb{R}^d)$ for $p > 2d/(d-2)$.

$(iv)$ For $d = 2$, give an example of an $H^1$ function which is not bounded.

**Exercice 4.9.** Let $r \in ]0,1[$. Show that $H^{\frac{d}{2}+r}(\mathbb{R}^d)$ is continuously embedded into the Hölder space $C^r(\mathbb{R}^d)$ defined in exercice 2.5.

**Exercice 4.10.** Let $s > \frac{d}{2}\cdot$ Show that there exists $C > 0$ such that

$$\forall u \in H^s(\mathbb{R}^d), \quad \|u\|_{L^\infty} \leq C\|u\|_{L^2}^{1-\frac{d}{2s}} \|u\|_{\dot{H}^s}^{\frac{d}{2s}}.$$

**Exercice 4.11.** Let $s > 0$ and $\Omega$ be a bounded open set of $\mathbb{R}^d$. Show that there exists $\lambda_s(\Omega) > 0$ such that

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \|\varphi\|^2_{\dot{H}^s(\mathbb{R}^d)} \geq \lambda_s(\Omega)\|\varphi\|^2_{L^2(\Omega)}.$$

**Exercice 4.12.** Let $\Omega$ be a open set of $\mathbb{R}^d$ of finite measure.

$(i)$ Show that the embedding $H^1_0(\Omega)$ into $L^2(\Omega)$ is compact.
Hint: use the Fourier transform to realize that if $(g_n)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$, then $(\widehat{g}_n)_{n\in\mathbb{N}}$ is bounded in the set of continuous functions which go to 0 as $|x| \to +\infty$.

$(ii)$ Prove that the Poincaré inequality still holds for this kind of domain.

$(iii)$ Show that

$$\forall u \in H^1(\Omega), \quad \|u - \overline{u}\|_{L^2} \leq C\|\nabla u\|_{L^2}$$

wher $\overline{u}$ is the average of $u$ on $\Omega$.

**Exercice 4.13.** Let $s \in ]0,1[$. Show that there exists $C > 0$ such that

$$\forall u \in \mathcal{S}, \quad C^{-1}\|u\|^2_{\dot{H}^s} \leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}}\, dx\, dy \leq C\|u\|^2_{\dot{H}^s}.$$

**Exercice 4.14.** Let $\chi \in \mathcal{S}(\mathbb{R}^d)$ and $s \in [0,1]$. Let the homogeneous *Fourier multiplier* $\widehat{|D|^s v} \stackrel{\text{def}}{=} |\xi|^s\widehat{v}$. Let the commutator

$$A_s v = [|D|^s, \chi] \stackrel{\text{def}}{==} |D|^s(\chi v) - \chi|D|^s v.$$

$(i)$ Let $v \in \mathcal{S}(\mathbb{R}^d)$, compute $\widehat{A_s v}$ in the form of an integral operator on $\widehat{v}$.

$(ii)$ Using Plancherel, show that $A_s$ is bounded on $L^2$.

$(iii)$ Give another proof of (iii) of Proposition 4.1.3.

**Exercice 4.15.** Let $u$ and $v$ in $\mathcal{S}(\mathbb{R}^d)$.

($i$) Compute $\|uv\|^2_{H^{s+t-\frac{d}{2}}}$ in terms of $\widehat{u}$ and $\widehat{v}$. Let

$$\begin{aligned} J_1 &= \int \langle\xi\rangle^{2s+2t-d}\left|\int_{2|\xi-\eta|\leq|\eta|}\widehat{u}(\xi-\eta)\widehat{v}(\eta)\,d\eta\right|^2 d\xi, \\ J_2 &= \int \langle\xi\rangle^{2s+2t-d}\left|\int_{\frac{|\eta|}{2}\leq|\xi-\eta|\leq|\eta|}\widehat{u}(\xi-\eta)\widehat{v}(\eta)\,d\eta\right|^2 d\xi \end{aligned}$$

with $\langle\xi\rangle := \sqrt{1+|\xi|^2}$.

($ii$) Assume $s < \frac{d}{2}$.

(a) Show that there exists $C = C(s,d)$ such that

$$\begin{aligned} \forall\xi\in\mathbb{R}^d, \quad \int_{2|\xi-\eta|\leq|\eta|}\left|\widehat{u}(\xi-\eta)\right|d\eta &\leq C\|u\|_{H^s}\langle\xi\rangle^{\frac{d}{2}-s}, \\ \forall\eta\in\mathbb{R}^d, \quad \int_{2|\xi-\eta|\leq|\eta|}\left|\widehat{u}(\xi-\eta)\right|d\xi &\leq C\|u\|_{H^s}\langle\eta\rangle^{\frac{d}{2}-s}. \end{aligned}$$

(b) Show that there exits $C' = C'(s,d)$ such that

$$J_1 \leq C'\|u\|^2_{H^s}\|v\|^2_{H^t}.$$

($iii$) We pick $(s,t)\in\mathbb{R}^2$ and assume $s+t>0$. Show that there eixst $C'' = C''(d,s,t)$ such that

$$J_2 \leq C''\|u\|^2_{H^s}\|v\|^2_{H^t}.$$

($iv$) Assume $s<\frac{d}{2}$, $t<\frac{d}{2}$ and $s+t>0$. Show that the multiplication operation $(u,v)\mapsto uv$ extends as a bilinear continuous map from $H^s\times H^t$ into $H^{s+t-\frac{d}{2}}$.

**Exercice 4.16.** The trace map. We define the trace map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{d-1})$ by

$$\tau u(x') = u(0,x'), \qquad x' = (x_2,\ldots,x_d).$$

($i$) Show that for all $u\in\mathcal{S}(\mathbb{R}^d)$ and $\xi'\in\mathbb{R}^{d-1}$,

$$\widehat{\tau u}(\xi') = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{u}(\xi_1,\xi')d\xi_1.$$

($ii$) Show that for $s>1/2$, $\exists C(s)>0$ such that $\forall u\in\mathcal{S}(\mathbb{R}^d)$,

$$\|\tau u\|_{H^{s-1/2}(\mathbb{R}^{d-1})} \leq C\|u\|_{H^s(\mathbb{R}^d)}.$$

Hint: use the previous question to derive the estimate

$$|\widehat{\tau u}(\xi')|^2 \leq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}}|\widehat{u}(\xi)|^2\langle\xi\rangle^{2s}d\xi_1\right)\left(\int_{\mathbb{R}}\langle\xi\rangle^{-2s}d\xi_1\right)$$

and express $\int_{\mathbb{R}}\langle\xi\rangle^{-2s}d\xi_1$ in terms of $\langle\xi'\rangle$ (where we noted $\xi=(\xi_1,\xi')$).

($iii$) Let $s>1/2$. Show that the trace application extends uniquely as a continuous map from $H^s(\mathbb{R}^d)$ onto $H^{s-1/2}(\mathbb{R}^{d-1})$.

(*iv*) Let $s > 1/2$ and $g \in H^{s-1/2}(\mathbb{R}^{d-1})$. Define

$$\widehat{v}(\xi) = \widehat{g}(\xi')\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}.$$

Show that $v \in H^s(\mathbb{R}^d)$ and $v(0,x') = Cg(x')$ for some constant $C \neq 0$. Conclude that the above trace map is surjective.

**Exercice 4.17.** [Sobolev space in a cube] Let $L^2_{per}(\mathbb{R}^d)$ be the set of functions $u : \mathbb{R}^d \to \mathbb{C}$ which are $2\pi\mathbb{Z}^d$ periodic and such that the restriction of $u$ to $Q_d \stackrel{\text{def}}{=} ]0,2\pi[^d$ belongs to $L^2(Q_d)$. We let $H^1_{per}(\mathbb{R}^d)$ the set of $u \in L^2_{per}(\mathbb{R}^d)$ such that $\nabla u \in L^2_{per}(\mathbb{R}^d)$. We equip $L^2_{per}(\mathbb{R}^d)$ and $H^1_{per}(\mathbb{R}^d)$ with the Hilbertian norms:

$$\|u\|_{L^2_{per}} \stackrel{\text{def}}{=} \sqrt{\frac{1}{(2\pi)^d}}\,\|u\|_{L^2(Q_d)} \quad \text{and} \quad \|u\|_{H^1_{per}} \stackrel{\text{def}}{=} \sqrt{\frac{1}{(2\pi)^d}\big(\|u\|^2_{L^2(Q_d)} + \|\nabla u\|^2_{L^2(Q_d)}\big)}\,.$$

For $k \in \mathbb{Z}^d$, we let $e_k(x) \stackrel{\text{def}}{=} e^{i(k|x)}$ and we define the discrete Fourier coefficients of $u$ by

$$u_k \stackrel{\text{def}}{=} \frac{1}{(2\pi)^d}\int_{Q_d} e^{-i(k|x)}u(x)\,dx.$$

(*i*) Compute $\|u\|_{L^2_{per}}$ and $\|u\|_{H^1_{per}}$ in terms of $u_k$.

(*ii*) Let $T_n : u \mapsto \sum_{|k|\leq n} u_k e_k$.

(a) Show that s $(T_n)_{n\in\mathbb{N}}$ converges to the map Id in $\mathcal{L}(H^1_{per}; L^2_{per})$.

(b) Conclude that the embedding $H^1_{per}(\mathbb{R}^d)$ into $L^2_{per}(\mathbb{R}^d)$ is compact.

**Exercice 4.18.** Let $1 \leq p < d$. We define the critical Sobolev exponent by $-s_c + \frac{d}{2} = -1 + \frac{d}{p}\cdot$

(*i*) For $p \geq 2$, show that $\forall s \geq s_c$, $H^s(\mathbb{R}^d) \hookrightarrow W^{1,p}(\mathbb{R}^d)$.

(*ii*) For $1 \leq p \leq 2$, show that $\forall s \leq s_c$, $W^{1,p}(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d)$.

**Exercice 4.19.** We propose a proof of Sobolev injection in dimension $d \geq 2$ as a consequence of the Hardy-Littlewood-Sobolev inequality.

(*i*) Show that there exists $C > 0$ such that for all $f \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$|f(x)| \leq C\int_{\mathbb{R}^d}\frac{|\nabla f(y)|}{|x-y|^{d-1}}\,dy.$$

Hint : notice that if $f$ is supported in the ball $B(0,R)$ then for all a $\omega \in \mathbb{S}^{d-1}$ the unit sphere, there holds

$$f(0) = -\int_0^R \frac{d}{dr}f(r\omega)\,dr.$$

(*ii*) Conclude that if $d < p < \infty$ then $W^{1,p}(\mathbb{R}^d)$ continuously embed into the space $\mathcal{C}_0(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$ which go to $0$ at infinity.

(*iii*) For $1 < p < d$, give a new proof of the critical Sobolev injection (4.29).

**Exercice 4.20.** Let $d = 3$.

$(i)$ Let $\psi(x) = \frac{1}{|x|}\cdot$ Show that $\widehat{\psi}$ is a homogeneous function of degre $-2$ with spherical symmetry. Show that

$$\widehat{\psi}(\xi) = \frac{c}{|\xi|^2}$$

for some $c \in \mathbb{R}^*$.
Hint: we recall that a distribution with support a singleton is a finite sum of Dirac masses (exercice 3.6).

$(ii)$ Recover using Theorem 4.2.2 the special case of the Hardy-Littlewood-Sobolev inequality:

$$\|\frac{1}{|x|} \star f\|_{L^6(\mathbb{R}^3)} \lesssim \|f\|_{L^{\frac{6}{5}}(\mathbb{R}^3)}.$$

# Chapter 5

# Scattering for the free Schrödinger semi group in $\mathbb{R}^d$

This chapter is devoted to the study of a central phenomenon in wave dynamics: scattering, that is the spreading of the energy of wave packets over all space. We shall restrict the exposition to the linear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u_{|t=0} = u_0 \end{cases}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}. \tag{5.1}$$

The spreading of the wave packet is due to the structure of the wave packet during the evolution: the speed of propagation in space of a wave packet localized at the frequency $\xi \in \mathbb{R}^d$ is an increasing fonction of $|\xi|$ as indicated by the dispersion relation [1] Hence the wave packet decomposes into elementary wave packets which travel at different speeds depending on their frequency: this is scattering.

The physical phenomenon is clear, but its actual translation into useful tools to the study the propagation, in particular for the study of nonlinear problems, has long been a challenge, especially for Schrödinger like models which produce infinite speed of propagation and little smoothing regularity. The pioneering works of R. Strichartz [38] at the end of the 1970's devoted to abstract harmonic analysis problems (restriction Theorems in Fourier analysis) have paved the way to the discovery of a new functional framework which has been developped for more than thirty years, and lead nowadays to breathrough results in the description of the long time behaviour of linear and non linear waves. The starting point is the series of works on the Cauchy problem for the nonlinear Schrödinger equation by Ginibre and Velo [16] which we will review in the next chapter.

Our aim in this chapter is to prove the celebrated Strichartz estimates and other dispersive properties of the flow in connection to the pseudo conformal symmetry.

## 5.1 The Schrödinger semi group in $\mathbb{R}^d$

We study in this section the linear flow (5.1) for an initial data $u_0 \in H^s(\mathbb{R}^d)$.

[1] computed by looking for a monochromatic wave $(t,x) \mapsto e^{i(\xi\cdot x - \omega t)}$ which through (5.1) yields $\omega = |\xi|^2$.

### 5.1.1 Representation formulas

The linear flow (5.1) is explicitely solvable in Fourier. In all the section, $\widehat{u}$ or $\mathcal{F}u$ is the Fourier transform in the space variables only, $t$ is seen as a parameter.

**Lemma** (Representation formula). *Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then the unique solution $u \in \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ to* (5.1) *is given by*

$$u(t,\cdot) = S(t)u_0 = S_t \star u_0 = \mathcal{F}^{-1}(e^{-it|\xi|^2}\widehat{u}_0(\xi)) \tag{5.2}$$

*with*

$$\forall t \in \mathbb{R}^*, \quad S_t \overset{def}{=} \frac{1}{(4\pi i t)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} \ \textit{and} \ \ S_0 \overset{def}{=} \delta_0.$$

*Proof of Lemma 5.1.1.* Let $u \in \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ solution to (5.1), then taking the Fourier transform in $x$ of (5.1) yields

$$\forall (t,\xi) \in \mathbb{R} \times \mathbb{R}^d, \ \ i\frac{d}{dt}\widehat{u}(t,\xi) - |\xi|^2\widehat{u}(t,\xi) = 0, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi),$$

which is explicitely integrated as

$$\forall (t,\xi) \in \mathbb{R} \times \mathbb{R}^d, \ \ \widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi),$$

and (5.2) is proved. The representation formula in space is a direct consequence of the computation of the Fourier transform of Gaussians which is proved below. □

**Lemma 5.1.1.** *Let $z \in \mathbb{C}$ which $\Re(z) > 0$, then*

$$\mathcal{F}\Big(e^{-z|\cdot|^2}\Big)(\xi) = \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

*with*[2] $z^{-\frac{d}{2}} \overset{def}{=} |z|^{-\frac{d}{2}} e^{-i\frac{d}{2}\theta}$ *for* $z = |z|e^{i\theta}$ *and* $\theta \in [-\pi/2, \pi/2]$.

*Proof of Lemma 5.1.1.* For all $\xi \in \mathbb{R}^d$, the functions

$$z \longmapsto \int_{\mathbb{R}^d} e^{-i(x|\xi)} e^{-z|x|^2}\, dx \quad \text{et} \quad z \longmapsto \Big(\frac{\pi}{z}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z}}$$

are holomorphic on $D = \{z \in \mathbb{C}, \Re(z) > 0\}$. The classical formula of Fourier transform for Gaussians ensures that these two functions coincide of the half real line $\{z = x > 0\}$, and hence on $D$. Let $t \neq 0$ and consider a sequence $(z_n)_{n\in\mathbb{N}}$ of $D$ converging to $it$. Thanks to Lebesgue dominated convergence theorem, for all $\phi \in \mathcal{S}$, there holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} e^{-z_n|x|^2}\phi(x)\, dx = \int_{\mathbb{R}^d} e^{-it|x|^2}\phi(x)\, dx \quad \text{and}$$
$$\lim_{n\to+\infty} \Big(\frac{\pi}{z_n}\Big)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4z_n}}\phi(\xi)\, d\xi = \Big(\frac{\pi}{it}\Big)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4it}}\phi(\xi)\, d\xi.$$

Since

$$\mathcal{F}\left(e^{-z_n|\cdot|^2}\right) = \Big(\frac{\pi}{z_n}\Big)^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4z_n}},$$

[2]choice of the determination of the complex logarithm.

we can write using the definiton of the Fourier transform for distributions (cf [4, 17]),

$$\langle \widehat{e^{-it|\cdot|^2}}, \phi\rangle = \langle e^{-it|\cdot|^2}, \widehat{\phi}\rangle = \lim_{n\to+\infty}\int e^{-z_n|x|^2}\widehat{\phi}(x)\,dx = \lim_{n\to+\infty}\int \widehat{e^{-z_n|\cdot|^2}}(\xi)\,\phi(\xi)\,d\xi$$

$$= \lim_{n\to+\infty}\left(\frac{\pi}{z_n}\right)^{\frac{d}{2}}\int e^{-\frac{|\xi|^2}{4z_n}}\phi(\xi)\,d\xi = \left(\frac{\pi}{it}\right)^{\frac{d}{2}}\int e^{-\frac{|\xi|^2}{4it}}\phi(\xi)\,d\xi,$$

which yields the expected equality for $z = it$. □

Sovling the homogeneous problem (5.1) allows one to solve the inhomogeneous problem.

**Lemma** (Duhamel formula)**.** *Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{C}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$, then the solution to $u \in \mathcal{C}^1(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$ of the inhomogeneous problem*

$$\begin{cases} i\partial_t u + \Delta u = f \\ u_{|t=0} = u_0 \end{cases} \tag{5.3}$$

*is given by Duhamel representation formula*

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')f(t')\,dt'. \tag{5.4}$$

*Proof of Lemma 5.1.1.* In Fourier, $u$ is a solution iff

$$\forall t \in \mathbb{R},\quad i\frac{d}{dt}\widehat{u}(t,\xi) - |\xi|^2\widehat{u}(t,\xi) = \widehat{f}(t,\xi),\quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi) \tag{5.5}$$

which is integrated explicitely as

$$\widehat{u}(t,\xi) = e^{-it|\xi|^2}\widehat{u}_0(\xi) - i\int_0^t e^{-i(t-t')|\xi|^2}\widehat{f}(t',\xi)\,dt',$$

and (5.4) follows through inverse Fourier transform. □

### 5.1.2 The Schrödinger semi group on $\mathbb{R}^d$

Observe that the representation formula (5.2) makes sense for $u_0 \in H^s(\mathbb{R}^d)$ and even $u_0 \in \mathcal{S}'(\mathbb{R}^d)$. The following Definition–Proposition is therefore an immediate consequence of (5.2) and Plancherel.

**Proposition** (Semi-group on $H^s(\mathbb{R}^d)$)**.** *Let $s \in \mathbb{R}$, we define for $u_0 \in H^s(\mathbb{R}^d)$ the Schrödinger semi group on $H^s$ by*

$$\forall t \in \mathbb{R},\quad S(t)u_0 = S_t \star u_0 = \mathcal{F}^{-1}(e^{-it|\xi|^2}\widehat{u}_0). \tag{5.6}$$

*Then $(S(t))_{t\in\mathbb{R}}$ is strongly continuous and unitary on $H^s$, ie:*
1.Regularity*: $t \mapsto S(t)u_0 \in \mathcal{C}(\mathbb{R}; H^s)$.*
2.$H^s$ isometric*: $\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s}$.*
3.Group property*: $\forall (t,t') \in \mathbb{R}^2$, $S(t)S(t')u_0 = S(t+t')u_0$ and $S(0) = \mathrm{Id}$.*
4. Adjoint*: $S(t)^* = S(-t)$ where the adjoint is with respect to the Hilbertian structure of $H^s$.*

Pointwise decay of the semi group follows directly from the representation formula (5.2).

**Proposition** (Pointwise decay). *Let $t \in \mathbb{R}^*$ and $p \in [2,+\infty]$, then $S(t)$ is strongly continuous from $L^{p'}$ into $L^p$ and*

$$\forall t \in \mathbb{R}^*, \quad \|S(t)u_0\|_{L^p} \leq \frac{1}{|4\pi t|^{\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})}}\|u_0\|_{L^{p'}}. \tag{5.7}$$

*Proof of Proposition 5.1.2.* Let $t \neq 0$, it suffice by density to establish (5.7) for $u_0 \in \mathcal{S}(\mathbb{R}^d)$, but then Young and (5.2) ensure:

$$\|S(t)u_0\|_{L^\infty} \leq \|u_0\|_{L^1}\|S_t\|_{L^\infty} \leq \frac{1}{|4\pi t|^{\frac{d}{2}}}\|u_0\|_{L^1}. \tag{5.8}$$

On the other hand, since the semi group is $L^2$ isometric:

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}.$$

Riesz-Thorin interpolation Theorem yields the claim. □

An immediate consequence of pointwise decay is the local decay of energy. Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$, then since $S(t)$ is unitary on $L^2$, the total mass is conserved:

$$\|S(t)u_0\|_{L^2} = \|u_0\|_{L^2}.$$

But the local in space mass is dissipated in time: let $R > 0$, then

$$\int_{|x|\leq R} |S(t)u_0|^2 \, dx \lesssim R^d \|S(t)u_0\|_{L^\infty}^2 \lesssim \frac{R^d}{|t|^d}\|u_0\|_{L^1}^2 \longrightarrow 0 \quad \text{as} \quad |t| \to +\infty.$$

We shall see, and this is a fundamental feature of dispersive problems, that the speed at which the local mass is dissipated is directly connected to the *regularity* of the data.

### 5.1.3 Weak solutions

The Schrödinger semi group (5.6) naturally extends to $\mathcal{S}'$, and then the equation (5.1) is satisfied *in the sense of Distributions*, [17].

**Definition 5.1.1** (Weak solution). *We say that a distribution $u \in \mathcal{C}(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ is a* weak solution *of the inhomogeneous problem* (5.3) *if for all $\varphi \in \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$, there holds*

$$\int_0^t \Big\langle u(t'), \Delta\varphi(t') - i\partial_t\varphi(t')\Big\rangle \, dt' = -i\big\langle u_0, i\varphi(0)\big\rangle + i\big\langle u(t), \varphi(t)\big\rangle + \int_0^t \langle f(t'), \varphi(t')\rangle \, dt'$$

*where $\langle\cdot,\cdot\rangle$ is the duality bracket of $\mathcal{S}'$ and $\mathcal{S}$.*

**Proposition 5.1.1** (The semi group produces weak solution). *Let $u_0 \in \mathcal{S}'$, then the distribution*

$$S(t)u_0 = \mathcal{F}^{-1}\Big(e^{-it|\xi|^2}\widehat{u}_0\Big) = S_t \star u_0 \quad \textit{with} \quad S_t(x) = \frac{1}{(4\pi i t)^{\frac{d}{2}}} e^{i\frac{|x|^2}{4t}} \tag{5.9}$$

*belongs to $\mathcal{C}^\infty(\mathbb{R}; \mathcal{S}')$ and is a weak solution to* (5.1).

**Remark 5.1.1** (Infinite speed of propagation). *The formula* (5.9) *shows an infinite speed of propagation property. Indeed, let $u_0 = \delta_{x=0}$ the Dirac mass at the origin, then* (5.9) *implies*

$$\forall t \neq 0, \quad u(t) = S_t,$$

*and hence $u(t)$ does not vanish at all on $\mathbb{R}^d$, even though the data was concentrated at the origin only.*

*Proof of Proposition 5.1.1.* Let $u(t) = \mathcal{F}^{-1}\Big(-e^{it|\xi|^2}\widehat{u}_0(\xi)\Big)$. For $\varphi \in \mathcal{C}^\infty(\mathbb{R};\mathcal{S})$, we note

$$I_\varphi(t) \stackrel{\text{def}}{=} \int_0^t \Big\langle u(t'), \Delta\varphi(t') - i\partial_t\varphi(t')\Big\rangle\, dt'.$$

By definition of $u$, we have

$$\begin{aligned} I_\varphi(t) &= \int_0^t \Big\langle \mathcal{F}^{-1}\left(e^{-it'|\xi|^2}\widehat{u}_0(\xi)\right), \Delta\varphi(t') - i\partial_t\varphi(t')\Big\rangle\, dt', \\ &= \int_0^t \Big\langle e^{-it'|\xi|^2}\widehat{u}_0(\xi), \mathcal{F}^{-1}\left(\Delta\varphi(t') - i\partial_t\varphi(t')\right)\Big\rangle\, dt', \\ &= -\int_0^t (2\pi)^{-d}\Big\langle \widehat{u}_0(\xi), e^{-it'|\xi|^2}\left(|\xi|^2\widehat{\varphi}(t',-\xi) + i\partial_t\widehat{\varphi}(t',-\xi)\right)\Big\rangle\, dt'. \end{aligned}$$

By definition of derivation in the sense of distributions:

$$I_\varphi(t) = -(2\pi)^{-d}\Big\langle \widehat{u}_0, \int_0^t e^{-it'|\xi|^2}\Big(|\xi|^2\widehat{\varphi}(t',-\xi) + i\partial_t\widehat{\varphi}(t',-\xi)\Big)\, dt'\Big\rangle.$$

Since

$$\partial_{t'}\left(e^{-it'|\xi|^2} i\widehat{\varphi}(t',-\xi)\right) = e^{-it'|\xi|^2}\Big(|\xi|^2\widehat{\varphi}(t',-\xi) + i\partial_{t'}\widehat{\varphi}(t',-\xi)\Big),$$

we obtain

$$\int_0^t e^{-it'|\xi|^2}\Big(|\xi|^2\widehat{\varphi}(t',-\xi) + i\partial_{t'}\widehat{\varphi}(t',-\xi)\Big)\, dt' = ie^{-it|\xi|^2}\widehat{\varphi}(t,-\xi) - i\widehat{\varphi}(0,-\xi).$$

Hence

$$\begin{aligned} I_\varphi(t) &= i(2\pi)^{-d}\langle \widehat{u}_0, e^{-it|\xi|^2}\widehat{\varphi}(t,-\xi)\rangle - i(2\pi)^{-d}\langle \widehat{u}_0, \widehat{\varphi}(0,-\xi)\rangle, \\ &= i\langle \widehat{u}(t), \mathcal{F}^{-1}\varphi(t)\rangle - i\langle \widehat{u}_0, \mathcal{F}^{-1}\varphi(0)\rangle = i\langle u(t), \varphi(t)\rangle - i\langle u_0, \varphi(0)\rangle \end{aligned}$$

and the claim is proved □

We may similarily extend the Duhamel formula which will be needed for the study of the non linear problem.

**Proposition 5.1.2** (Low regularity Duhamel formula)**.** *Let* $u_0 \in L^2$ *and* $f \in L^1_{loc}(\mathbb{R};L^2)$ *then the free Schrödinger equation* (5.3) *has a* unique *weak solution* $u \in \mathcal{C}(\mathbb{R};L^2)$ *which is given by the Duhamel formula* (5.4)*. Moreover, the mass evolves according to:*

$$\forall t \in \mathbb{R}, \quad \|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2} + 2\Im m\int_0^t \int_{\mathbb{R}^d} f(\tau,x)\bar{u}(\tau,x)\, dx\, d\tau. \tag{5.10}$$

*Proof of Proposition 5.1.2.* Assume first $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and $f \in \mathcal{C}(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$, then $u \in \mathcal{C}^1(\mathbb{R};\mathcal{S}(\mathbb{R}^d))$ and we may take $\bar{u}$ as a test function in the definition of the weak formulation. We obtain after division by $i$:

$$\|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2} + i\int_0^t \int_{\mathbb{R}^d} f\bar{u}\, dx\, d\tau - i\int_0^t \int_{\mathbb{R}^d} u(\Delta\bar{u} - i\partial_t\bar{u})\, dx\, d\tau. \tag{5.11}$$

We have $-i\partial_t\bar{u} + \Delta\bar{u} = \bar{f}$ and the result follows. To prove (5.10) for data $u_0 \in L^2$ and $f \in L^1_{loc}(\mathbb{R};L^2)$, we regularize : using the convolution by an approximation of identity in $t$, we construct $u_0^n \in \mathcal{S}$ with $u_0^n \to u$ in $L^2$ and $f^n \in \mathcal{S}$ with $f^n \to f$ in $L^1_{loc}(\mathbb{R}; L^2)$. Let $u^n \in \mathcal{C}^1(\mathbb{R}; \mathcal{S})$ be the solution associated to $(u_0^n, f^n)$ by Duhamel, then for all $(n,m) \in \mathbb{N}^2$,

$$i\partial_t(u^n - u^m) + \Delta(u^n - u^m) = f^n - f^m \quad \text{et} \quad (u^n - u^m)_{|t=0} = u_0^n - u_0^m.$$

Hence

$$\|(u^n - u^m)(t)\|_{L^2} \leq \|u_0^n - u_0^m\|_{L^2} + \left| \int_0^t \|f^n - f^m\|_{L^2}\, d\tau \right|.$$

and hence $(u^n)_{n\in\mathbb{N}}$ converges strongly in $\mathcal{C}(\mathbb{R}; L^2)$. We may therefore pass to the limit in the weak formulation and ensures that (5.10) is still valid for the limit $u$ of this sequence. Finally, uniqueness if obtained by remarking that if $u \in \mathcal{C}(\mathbb{R}; L^2)$ is a solution to (5.1) with $f \equiv 0$ and $u_0 \equiv 0$, then so is $u^n \stackrel{\text{def}}{=} \chi_n \star u$ with $\chi_n(t,x) \stackrel{\text{def}}{=} n^{1+d}\chi(nt, nx)$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^{1+d})$ of integral 1. We have $u^n \in \mathcal{C}^\infty(\mathbb{R}; H^\infty)$ with $H^\infty(\mathbb{R}^d) \stackrel{\text{def}}{=} \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d)$. Passing in Fourier, we see that a.s. in $\xi \in \mathbb{R}^d$, the smooth function $t \mapsto \widehat{u}^n(t,\xi)$ satisfies the differential equation (5.5) with source term and data 0, and hence $\widehat{u}^n \equiv 0$. We conclude $\widehat{u} \equiv 0$ by passing to the limit, and hence $u \equiv 0$. □

## 5.2 Strichartz space-time bounds

We give in this section a self contained proof of Strichartz estimates which yield an improved regularity *in space* of $u(t,\cdot)$ provided a suitable averaging process in time. The corresponding estimates are the corner stone to the resolution of the nonlinear Cauchy problem and the derivation of the long time asymptotics of the flow. More precisely, we will transform the pointwise in time estimate (5.7) into an averaged temporal bound of the type

$$\|S(t)u_0\|_{L^q_t L^r_x} \leq C\|u_0\|_{L^2} \tag{5.12}$$

where

$$\|u\|_{L^q_t L^r_x} = \left| \begin{array}{l} \left(\int_{\mathbb{R}} \|u(t,\cdot)\|^q_{L^r_x}\, dt\right)^{\frac{1}{q}} \text{ for } 1 \leq q < +\infty \\ \|u\|_{L^\infty_t L^r_x} = \sup_{t\in\mathbb{R}} \|u(t,\cdot)\|_{L^r_x} \text{ for } q = +\infty \end{array} \right.$$

**Remark 5.2.1** (Scale invariant estimate)**.** *The estimate* (5.12) *is scale invariant. Indeed, let* $\lambda \in \mathbb{R}^*$*, and* $u_\lambda(x) = u(\lambda x)$ *then an explicit computation ensures*

$$S(t)u_\lambda = (S(\lambda^2 t)u)_\lambda.$$

*This implies immediately that it if* (5.12) *holds, then necessarily* $(q,r)$ *must satisfy the compatibility relation:*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}\cdot \tag{5.13}$$

**Remark 5.2.2** (Gain of regularity)**.** *The estimate* (5.12) *is an improvement with respect to Sobolev embeddings. Indeed, since the semi group is unitary on* $H^s$*, then* $u_0 \in H^s$ *implies* $u(t) \in H^s$*. To obtain* $L^r$ *control through Sobolev requires* $s = d/2 - d/r$*. The estimate* (5.12) *shows that* $u(t) = S(t)u_0 \in L^r$ *for a.e.* $t$ *even if* $u_0 \in L^2$*. Note that the gain of regularity is a.e. in time and hence does not contradict the time reversibility of the flow.*

**Definition 5.2.1** (Admissible pair)**.** *We say* $(q,r) \in [2,\infty]^2$ *is* admissible *if* (5.13) *holds and* $(q,r,d) \neq (2,\infty,2)$*. We say that is strictly admissible if moreover*[3] $(q,r) \neq (2, \frac{2d}{d-2})\cdot$

[3]these are the "endpoints"

**Theorem** (Strichartz estimates). *Let $d \geq 1$.*
*(i)* Homogeneous case*: for all admissible pair $(q, r)$, there exists $C$ such that for all $u_0 \in L^2(\mathbb{R}^d)$, the solution $S(t)u_0 \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ to the homogeneous problem*

$$\begin{cases} i\partial_t u + \Delta u = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{5.14}$$

*satisfies*

$$\|S(t)u_0\|_{L^q_t(L^r_x)} \leq C\|u_0\|_{L^2}. \tag{5.15}$$

*(ii)* Inomogeneous case*: for every admissible pairs $(q_1, r_1), (q_2, r_2)$, there exists $C$ such that for all $f \in L^{q_2'}(\mathbb{R}; L^{r_2'}(\mathbb{R}^d))$, the unique solution $v \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^d))$ given by the Duhamel formula to*

$$\begin{cases} i\partial_t u + \Delta u = f, \\ u_{|t=0} = 0 \end{cases} \tag{5.16}$$

*satisfies*

$$\|u\|_{L^{q_1}_t(L^{r_1}_x)} \leq C\|f\|_{L^{q_2'}_t(L^{r_2'}_x)}. \tag{5.17}$$

*Proof of Theorem 5.2.* We treat the case of admissible pair only and refer to the seminal paper of M. Keel and T. Tao [20] or [2] for the endpoint case

$$(q, r) = \left(2, \frac{2d}{d-2}\right) \quad \text{and} \quad d \geq 3.$$

By density, we may assume all functions are smooth and decaying at $\infty$.

**step 1** The $TT^\star$ lemma. Let us start with an abstract elementary lemma.

**Lemma** ($TT^\star$)**.** *Let $T \in \mathcal{L}(\mathcal{H}, B)$ where $\mathcal{H}$ is Hilbert and $B$ Banach, and let $T^\star : B' \to \mathcal{H}$ be the adjoint operator defined by:*

$$(T^\star x|y)_{\mathcal{H}} = \langle x, \overline{Ty}\rangle_{B' \times B}.$$

*Then:*

$$\|TT^\star\|_{\mathcal{L}(B';B)} = \|T\|^2_{\mathcal{L}(\mathcal{H};B)} = \|T^\star\|^2_{\mathcal{L}(B';\mathcal{H})}. \tag{5.18}$$

**Remark 5.2.3.** *In other words, it is equivalent to show that $T \in \mathcal{L}(\mathcal{H}; B)$, $T^\star \in \mathcal{L}(B'; \mathcal{H})$ or $TT^\star \in \mathcal{L}(B'; B)$, and then* (5.18) *holds.*

*Proof of Lemma 5.2.* By characterization of the norm in a Hilbert space:

$$\|T^\star x\|_{\mathcal{H}} = \sup_{\|y\|_{\mathcal{H}}=1} |(T^\star x|y)_{\mathcal{H}}|.$$

Hence

$$\|T^\star x\|_{\mathcal{H}} = \sup_{\|y\|_{\mathcal{H}}=1} |\langle x, \overline{Ty}\rangle_{B' \times B}| \leq \|x\|_{B'} \sup_{\|y\|_{\mathcal{H}}=1} \|Ty\|_{\mathcal{B}} \leq \|T\|_{\mathcal{L}(\mathcal{H};B)}\|x\|_{\mathcal{B}'}$$

and thus $\|T^\star\|_{\mathcal{L}(B';\mathcal{H})} \leq \|T\|_{\mathcal{L}(\mathcal{H};B)}$. Similarly $\|T\|_{\mathcal{L}(\mathcal{H};B)} \leq \|T^\star\|_{\mathcal{L}(B';\mathcal{H})}$ and then $\|TT^\star\|_{\mathcal{L}(B';B)} \leq \|T\|_{\mathcal{L}(\mathcal{H};B)}\|T^\star\|_{\mathcal{L}(B';\mathcal{H})}$ by composition. Using the Hilbertian structure again:

$$\|T^\star x\|^2_{\mathcal{H}} = (T^\star x|T^\star x)_{\mathcal{H}} = \langle x, \overline{TT^\star x}\rangle_{B' \times B} \leq \|x\|^2_{B'}\|TT^\star\|_{\mathcal{L}(B';B)}$$

and hence $\|T^\star\|^2_{\mathcal{L}(B';\mathcal{H})} \leq \|TT^\star\|_{\mathcal{L}(B';B)}$ and (5.18) is proved. $\square$

**step 2** Computing $T$, $T^\star$ and $TT^\star$. Let $(q,r)$ be strictly admissible. We apply the $TT^\star$ lemma[4] to

$$\mathcal{H}=L^2(\mathbb{R}^d),\quad B=L^q(\mathbb{R};L^r(\mathbb{R}^d)),\quad B'=L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^d))$$

and

$$T:u_0\longmapsto\big[t\mapsto S(t)u_0\big].$$

Observe that since $S^\star(t)=S(-t)$

$$\begin{aligned}\langle g,\overline{Tu_0}\rangle_{B'\times B} &= \int_{\mathbb{R}\times\mathbb{R}^d} g(t,x)\overline{S(t)u_0(x)}\,dx\,dt=\int_{\mathbb{R}}(g(t,\cdot)|S(t)u_0)_{\mathcal{H}}\,dt\\ &= \int_{\mathbb{R}}(S(-t)g(t,\cdot)|u_0)_{\mathcal{H}}\,dt=\left(\int_{\mathbb{R}}S(-t)g(t,\cdot)\,dt\middle|u_0\right)_{\mathcal{H}}\end{aligned}$$

and hence the computation of the adjoint:

$$T^\star:\varphi\longmapsto\int_{\mathbb{R}}S(-t')\varphi(t',\cdot)\,dt'\quad\text{et}\quad TT^\star:\varphi\longmapsto\left[t\mapsto\int_{\mathbb{R}}S(t-t')\varphi(t',\cdot)\,dt'\right].$$

In particular, up to the domain of integration in time, $TT^*f$ is very close to the Duhamel term (5.4) assocated to the inhomogeneous equation (5.16).

**step 3** Homogeneous estimate. The key idea is to estimate the norm of $TT^\star$ instead of $T$, using pointwise decay (5.7) as follows :

$$\begin{aligned}\|TT^\star g(t,\cdot)\|_{L^r_x} &= \left\|\int_{\mathbb{R}}S(t-t')g(t',\cdot)\,dt'\right\|_{L^r_x}\le\int_{\mathbb{R}}\|S(t-t')g(t',\cdot)\|_{L^r_x}\,dt'\\ \lesssim\ \int_{\mathbb{R}}\frac{1}{|t-t'|^{\frac{d}{2}(\frac{1}{r'}-\frac{1}{r})}}\|g(t',\cdot)\|_{L^{r'}_x}\,dt' &= \int_{\mathbb{R}}\frac{1}{|t-t'|^{\frac{2}{q}}}\|g(t',\cdot)\|_{L^{r'}_x}\,dt'\end{aligned}$$

where we used the admissible pair relation:

$$\frac{d}{2}\left(\frac{1}{r'}-\frac{1}{r}\right)=\frac{d}{2}\left(1-\frac{2}{r}\right)=\frac{2}{q}\cdot$$

We apply the one dimensional Hardy-Littlewood-Sobolev estimate in time: for $0<2/q<1$,

$$\left\|\frac{1}{|t|^{\frac{2}{q}}}\star h\right\|_{L^q_t}\lesssim\|h\|_{L^\gamma_t}\quad\text{avec}\quad 1+\frac{1}{q}=\frac{1}{\gamma}+\frac{2}{q}\cdot$$

Hence $\gamma=q'$ and

$$\|TT^\star g\|_{L^q_tL^r_x}\lesssim\left\|\frac{1}{|t|^{\frac{2}{q}}}\|g(t,\cdot)\|_{L^{r'}_x}\right\|_{L^{\frac{2}{q}}_t}\lesssim\|g\|_{L^{q'}_tL^{r'}_x},$$

and hence using the $TT^\star$ Lemma:

$$\|T\|^2_{\mathcal{L}(\mathcal{H};B)}=\|T^\star\|^2_{\mathcal{L}(B';\mathcal{H})}=\|TT^\star\|_{\mathcal{L}(B';B)}<\infty\tag{5.19}$$

which concludes the proof of the homogeneous estimate (5.15) in the case $2<q<\infty$. The case $q=\infty$ is $r=2$ and this is just conservation of mass for the free Schrödinger group.

[4]it suffice as usual to argue for Schwartz functions for which the definition of $T$ and $T^*$ make perfect sense, only the norms in which we estimate these terms matter.

**step 4** Inhomogeneous estimate. The proof of the inhomogeneous estimate (5.17) when $(q_1, r_1) = (q_2, r_2) = (q, r)$ is similar to step 3 since the formula (5.4) with $u_0 = 0$ is $TT^*$ with integration restricted to the interval $[0, t]$. Indeed, the solution $v$ to the inhomogeneous problem (5.16) satisfies

$$v(t, \cdot) = -i \int_{\mathbb{R}} \chi(t, t') S(t - t') f(t', \cdot)\, dt'$$

with

$$\chi(t, t') \stackrel{\text{def}}{=} \begin{cases} 1 & \text{si} \quad 0 \leq t' \leq t \quad \text{ou} \quad t \leq t' \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\chi$ is bounded by 1, we can write

$$\|v(t, \cdot)\|_{L^r_x} \leq \int_0^t \|S(t - t') f(t', \cdot)\|_{L^r_x}\, ds$$

and hence as above pointwise decay and Hardy-Littlewood-Sobolev ensure

$$\|v\|_{L^q_t L^r_x} \lesssim \left\| \int_0^t \frac{1}{|t - t'|^{\frac{2}{q}}} \|f(t, \cdot)\|_{L^{r'}_x}\, dt' \right\|_{L^q_t} \lesssim \left\| \frac{1}{|t|^{\frac{2}{q}}} \star \|f(t, \cdot)\|_{L^{r'}_x} \right\|_{L^q_t} \lesssim \|f\|_{L^{q'}_t L^{r'}_x}.$$

Let us now prove

$$\|v\|_{L^\infty_t L^2_x} \lesssim \|f\|_{L^{q_2'}_t L^{r_2'}_x}. \tag{5.20}$$

For this, the group structure of $S(t)$ yields

$$\begin{aligned} v(t, \cdot) &= -i \int_{\mathbb{R}} \chi(t, t') S(t - t') f(t', \cdot)\, dt' = -iS(t) \int_{\mathbb{R}} \chi(t, t') S(-t') f(t', \cdot)\, dt' \\ &= -iS(t) T^\star (\chi(t, \cdot) f) \end{aligned}$$

and hence using the $L^2$ bound for $S(t)$ and remarking that (5.19) (i.e. $T : L^2 \to L^{q_2}_t(L^{r_2}_x)$ ensures $T^\star : L^{q_2'}_t(L^{r_2'}_x) \to L^2$), we obtain for $t \in \mathbb{R}$ for $(q_2, r_2)$ admissible,

$$\|v(t, \cdot)\|_{L^2_x} = \|T^* (\chi(t, \cdot) f)\|_{L^2_x} \lesssim \|\chi(t, \cdot) f\|_{L^{q_2'}_t L^{r_2'}_x} \lesssim \|f\|_{L^{q_2'}_t L^{r_2'}_x},$$

and (5.20) is proved.

The linear map $U : f \mapsto v$ is bounded from $L^{q_2'}_t L^{r_2'}_x$ into $L^\infty_t L^2_x \cap L^{q_2}_t L^{r_2}_x$, and hence using the Riesz-Thorin interpolation Theorem generalized to space time Lebesgue (cf Theorem 1.2.3) ensures that it also bounded from $L^{q_2'}_t L^{r_2'}_x$ into $L^{q_1}_t L^{r_1}_x$ for all admissible pair $(q_1, r_1)$, $(q_2, r_2)$ with $q_2 \leq q_1 \leq +\infty$. The case $q_2 \geq q_1$ follows by duality. More precisely, assume

$$U \text{ is bounded from } L^1_t L^2_x \text{ into } L^{q_1}_t L^{r_1}_x \tag{5.21}$$

for all $(q_1, r_1)$ strictly admissible, then since $U$ is also bounded from $L^{q_1'}_t L^{r_1'}_x$ into $L^{q_1}_t L^{r_1}_x$, we obtain (5.17) for all strictly admmissible pair $(q_1, r_1)$, $(q_2, r_2)$ such that $q_1 \leq q_2 \leq +\infty$. To prove (5.21), we write (simple generalization of Lemma 2.2.4) :

$$\|v\|_{L^{q_1}_t L^{r_1}_x} = \sup_{\|\phi\|_{L^{q_1'}_t L^{r_1'}_x} \leq 1} \left| \int_{\mathbb{R} \times \mathbb{R}^d} v\, \bar{\phi}\, dx\, dt \right|.$$

By density, we may assume functions are smooth and decaying at infinity. Letting $v = Uf$ and recalling the definition of $U$ yields

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}^d} v\,\bar\phi\,dx\,dt &= \int_{\mathbb{R}\times\mathbb{R}^d}\int_0^t S(t-t')\,f(t')\,\bar\phi(t)\,dt'\,dt\,dx\\ &= \int_{\mathbb{R}}\int_0^t \big(S(t)S(-t')f(t')|\phi(t)\big)_{L^2(\mathbb{R}^d)}\,dt'\,dt\\ &= \int_{\mathbb{R}}\left(S(-t')f(t')\Big|\int_{\mathbb{R}} S(-t)\chi(t,t')\phi(t)\,dt\right)dt'\end{aligned}$$

with $\chi$ defined above. By Cauchy-Schwarz, we conclude

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}^d} v\,\bar\phi\,dx\,dt\right| \le \int_{\mathbb{R}} \|S(-t')f(t')\|_{L^2(\mathbb{R}^d)}\,\|T^*(\chi(\cdot,t')\phi)\|_{L^2(\mathbb{R}^d)}\,dt'.$$

But $T : L^2 \to L_t^{q_1}L_x^{r_1}$ implies $T^* : L_t^{q_1'}L_x^{r_1'} \to L^2$, and hence for all $t' \in \mathbb{R}$,

$$\|T^*(\chi(\cdot,t')\phi)\|_{L^2(\mathbb{R}^d)} \lesssim \|\chi(\cdot,t')\phi\|_{L_t^{q_1'}L_x^{r_1'}} \le \|\phi\|_{L_t^{q_1'}L_x^{r_1'}},$$

and hence since $S(-t')$ is unitary on $L^2(\mathbb{R}^d)$,

$$\left|\int_{\mathbb{R}}\int_{\mathbb{R}^d} v\,\bar\phi\,dx\,dt\right| \lesssim \|f\|_{L_t^1L_x^2}\|\phi\|_{L_t^{q_1'}L_x^{r_1'}}.$$

This conclude the proof of Theorem 5.2 for strictly admissible pairs. □

**Remark 5.2.4.** *The limit cases $q = 2$ and $r = 2d/(d-2)$ with $d \ge 3$ are more delicate to handle, we refer to [20] or [2] for a slightly different proof relies on an atomic decomposition analogous to the one used in Chapter 1).*

## 5.3 Local in space decay in weighted spaces

The pointwise decay estimate (5.8) is optimal in the sense of the norms used, but it does give a very clear description of the dispersion mechanism. The spreading of the wave packet can be seen directly for Schödinger using the pseudo conformal symmetry which requires data in the virial space

$$\Sigma \stackrel{\text{def}}{=} \big\{u \in H^1 \,:\, xu \in L^2\big\}. \tag{5.22}$$

We shall admit the following elementary result ( see for example [7]) which follows by a regularization argument :

**Theoreme 5.3.1.** *$S(t)$ is strongly continuous on $\Sigma$.*

A fundamental algebraic fact is the existence of an explicit pseudo conformal symmetry for the linear Schrödinger flow.

**Proposition** (Conformal invariance)**.** *Let $v$ solve (5.1) then so does*

$$u(t,x) \stackrel{def}{=} \frac{1}{(1+t)^{\frac d2}}v\left(\frac{t}{1+t},\frac{x}{1+t}\right)e^{i\frac{|x|^2}{4(1+t)}} \tag{5.23}$$

*for all $t \neq -1$.*

*Proof of Proposition 5.3.* Let $u$ solve (5.1), and consider the renormalization

$$u(t,x)=\frac{1}{\lambda(t)^{\frac{d}{2}}}w(s,y)\ \text{ avec }\ \frac{ds}{dt}=\frac{1}{\lambda^2(t)}\ \text{ et }\ y=\frac{x}{\lambda(t)}\cdot \tag{5.24}$$

for some scaling factor $\lambda(t)>0$ to be determined. Let $\lambda_s=\frac{d\lambda}{ds}$, we compute:

$$0=(i\partial_t u+\Delta u)(t,x)=\frac{1}{\lambda^{2+\frac{d}{2}}}\left(i\partial_s w-i\frac{\lambda_s}{\lambda}\left[\frac{d}{2}w+y\cdot\nabla w\right]+\Delta w\right)(s,y).$$

We now map the above linear operator onto the harmonic oscillator as follows: let

$$w(s,y)=v(s,y)e^{-i\frac{b(s)|y|^2}{4}}\ \text{ with }\ b=-\frac{\lambda_s}{\lambda},$$

then

$$i\partial_s v+\Delta v+\left(\frac{b_s+b^2}{4}\right)|y|^2v=0. \tag{5.25}$$

An explicit symmetry of the linear Schrödinger equation is therefore provided by the choice

$$\begin{cases}\frac{\lambda_s}{\lambda}=-b,\\ b_s+b^2=0,\\ \frac{ds}{dt}=\frac{1}{\lambda^2}.\end{cases} \tag{5.26}$$

To integrate (5.26), we compute

$$\left(\frac{b}{\lambda}\right)_s=\frac{b_s}{\lambda}-\frac{b\lambda_s}{\lambda^2}=\frac{b_s+b^2}{\lambda}=0$$

and hence

$$b=-\frac{\lambda_s}{\lambda}=-\lambda\lambda_t=c\lambda.$$

The choice $c=-1$ (i.e. $\lambda(0)=1$ and $\lambda_t(0)=1$) gives

$$\lambda(t)=1+t,\quad b(t)=-\lambda(t)=-(1+t)$$

and hence choosing $s(0)=0$:

$$s(t)=\int_0^t\frac{d\tau}{\lambda^2(\tau)}=\frac{t}{1+t},$$

this is (5.23). □

A spectacular consequence of (5.23) is the complete description in the physical space of the dispersion of the wave packet.

**Proposition** (Dispersion in $\Sigma$)**.** *Let $u_0\in\Sigma$ and $u$ the corresponding solution to* (5.1) *given by Theorem 5.3.1. Then there exists $u^*\in\Sigma$ such that*

$$\left\|u(t,\cdot)-\frac{1}{|t|^{\frac{d}{2}}}\left(u^*e^{it\frac{|y|^2}{4}}\right)\left(\frac{\cdot}{t}\right)\right\|_{L^2}\longrightarrow 0\quad quand\quad t\to+\infty. \tag{5.27}$$

*Proof of Proposition 5.3.* Let $v$ be given by (5.23). Then $v$ satisfies (5.1) and

$$s(t) = \frac{t}{1+t} \to 1 \ \text{ as } \ t \to +\infty.$$

By $L^2$ continuity of $S(t)$, we conclude

$$v(s,y) \to v(1,y) \ \text{ in } \ L^2 \ \text{ as } \ s \to 1.$$

Let $u^\star(y) \overset{\text{def}}{=} v(1,y)$ and $y = x/(1+t)$, we conclude

$$\frac{1}{(1+t)^{\frac{d}{2}}} \left\| v\Big(\frac{t}{1+t}, \frac{x}{1+t}\Big) - u^\star\Big(\frac{x}{1+t}\Big) \right\|_{L^2} \to 0 \text{ as } \ t \to +\infty,$$

which using (5.23) implies (5.27). □

In other words, the spreading of the wave packet induced by the physical separation of wave packets of different frequencies yields in physical space a profile that spreads in space at a *universal* speed vitesse $\lambda(t) \sim t$ modulo an explicit quadratic oscillation. A corollary is an improved decay estimate in weighted spaces.

**Proposition** (Local energy decay in weighted space)**.** *For $s \geq 0$, we define the operator* multiplicateur de Fourier $|D|^s$ *by*

$$\mathcal{F}(|D|^s v) \overset{def}{=} |\xi|^s \mathcal{F} v.$$

*Then for all $u$ solving* (5.1) *in* $\mathcal{C}^1(\mathbb{R};\mathcal{S})$, *there holds :*

$$\left\| |D|^s \left( u e^{-i\frac{|x|^2}{4(1+t)}} \right) \right\|_{L^2} \leq C_s \frac{\left\| |D|^s \left( u_0 e^{-i\frac{|x|^2}{4}} \right) \right\|_{L^2}}{(1+t)^s}. \tag{5.28}$$

*Proof of Proposition 5.3.* Recall (5.23). Since $v$ solves (5.1), the conservation of the norms $\|\cdot\|_{\dot{H}^s}$ through the action of $S(t)$ ensures

$$\||D|^s z(\lambda^{-1}\cdot)\|_{L^2} = \|z(\lambda^{-1}\cdot)\|_{\dot{H}^s} = \lambda^{\frac{d}{2}-s}\|z\|_{\dot{H}^s} = \lambda^{\frac{d}{2}-s}\||D|^s z\|_{L^2}$$

and hence

$$\left\| |D|^s \left( u e^{-i\frac{|x|^2}{4(1+t)}} \right) \right\|_{L^2} = \frac{1}{(1+t)^s} \left\| |D|^s v\left(\frac{t}{1+t}, \cdot\right) \right\|_{L^2} = \frac{1}{(1+t)^s} \||D|^s v\,(0,\cdot)\|_{L^2}.$$

It remains to replace $v(0,\cdot)$ by its value and the claim is proved. □

**Remark 5.3.1.** *Combining the above Proposition with the Gagliardo-Nirenberg estimate (see exercice 4.10) :*

$$\|U\|_{L^\infty} \lesssim \||D|^s U\|_{L^2}^{\frac{d}{2s}} \|U\|_{L^2}^{1-\frac{d}{2s}}$$

*applied with* $U = u e^{-i\frac{|x|^2}{4(1+t)}}$, *retrieves* (5.7), *but with a different norm on the data*

Compared to (5.7), the estimate (5.28) expresses an improved local enegy decay for higher Sobolev norms modulo the quadratic phase, but to the expense of higher control of the data. The method presented here for the Schrödinger equation is a canonical application of *Klainerman's vector field method*: decay in time is proved by writing down the conservation laws for suitable transformations of the solution related to the symmetry group, and this method has extremely deep ramifications in particular for the study of the quasilinear waves of general relativity, see [8].

## 5.4 Exercices

**Exercice 5.1** (Dispersion for the free transport)**.** Let the transport equation describing the evolution of the microscopic density $f(t,x,v)\in\mathbb{R}^+$ of free particules which are at $x\in\mathbb{R}^d$ with the speed $v\in\mathbb{R}^d$ at time $t\in\mathbb{R}$ :

$$(T)\qquad \begin{cases} \partial_t f + v\cdot\nabla_x f = 0,\\ f_{|t=0} = f_0.\end{cases}$$

(*i*) Assume $f_0 = f_0(x,v)$ is differentiable, compute the solution to $(T)$.

(*ii*) If $f_0$ is moreover integrable, show that the total density is converved

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\,dx\,dv.$$

(*iii*) We define the macroscopic density $\rho(t,x)\overset{\text{def}}{=}\int_{\mathbb{R}^d} f(t,x,v)\,dv$. Show the pointwise decay:

$$\|\rho(t,\cdot)\|_{L^\infty}\le \frac{1}{|t|^d}\|\sup_v f_0(\cdot,v)\|_{L^1}\quad \text{for all } t\neq 0.$$

**Exercice 5.2** (Wave equation)**.** Let the free wave equation

$$(W)\qquad \begin{cases} \Box u = 0\\ (u,\partial_t u)_{|t=0} = (u_0,u_1)\end{cases}$$

where $\Box\overset{\text{def}}{=}\partial_t^2-\Delta$ and where $u=u(t,x)\in\mathbb{R},\ (t,x)\in\mathbb{R}\times\mathbb{R}^d$.

(*i*) For $d=1$ and $(u_0,u_1)\in C^2\times C^1$, show that the $C^2$ solution is given by *d'Alembert's formula*:

$$u(t,x)=\frac12\Big(u_0(x+t)+u_0(x-t)+\int_{x-t}^{x+t}u_1(y)\,dy\Big).$$

(*ii*) For $d=3$, we recall that the solution is given by

$$u(t,x)=\frac{1}{4\pi}\Big(\frac1t\int_{S(x,t)}u_1(\sigma)\,d\sigma+\frac{d}{dt}\Big(\frac1t\int_{S(x,t)}u_0(\sigma)\,d\sigma\Big)\Big)$$

wher $S(x,t)$ is the sphere of center $x$ and radius $t$. Assume for simplicity $u_0\equiv 0$, then show:

$$\|u(t)\|_{L^\infty}\lesssim \frac{\|\nabla u_1\|_{L^1}}{|t|}+\frac{\|u_1\|_{L^1}}{t^2}.$$

**Exercice 5.3** (Oscillatory integrals)**.** Let $a\in\mathcal{D}(\mathbb{R})$ and $\Phi$ a $C^2$ function such that for some $c_0>0$:

$$\forall x\in \text{Supp }a,\ \Phi''(x)\ge c_0.$$

For $t\in\mathbb{R}$, we define the oscillatory integral

$$I(t)\overset{\text{def}}{=}\int_{\mathbb{R}} e^{it\Phi(x)}a(x)\,dx.$$

For $t\neq 0$, we define the differential operator $\mathcal{L}_t$ acting on derivable functions $b$ by

$$\mathcal{L}_t b(x)\overset{\text{def}}{=}\frac{1}{1+t\big(\Phi'(x)\big)^2}\big(b(x)-i\Phi'(x)b'(x)\big).$$

$(i)$ Using $\mathcal{L}_t$, show that $I(t) = I_1(t) + I_2(t)$ with

$$I_1(t) \stackrel{\text{def}}{=} \int e^{it\Phi(x)} \frac{i\Phi'(x)}{1+t(\Phi'(x))^2} a'(x)\,dx \quad \text{and}$$

$$I_2(t) \stackrel{\text{def}}{=} \int \frac{e^{it\Phi(x)}}{1+t(\Phi'(x))^2}\Big(1 + i\Phi''(x) - 2i\frac{t(\Phi'(x))^2\Phi''(x)}{1+t(\Phi'(x))^2}\Big)a(x)\,dx.$$

$(ii)$ Noticing that for $x \in \operatorname{Supp}\, a$,

$$\frac{1}{1+t(\Phi'(x))^2} \leq \frac{1}{c_0}\frac{\Phi''(x)}{1+t(\Phi'(x))^2},$$

show that

$$|I_2(t)| \leq \frac{\pi}{2}\left(\frac{1}{c_0} + 3\right)\frac{1}{|t|^{\frac{1}{2}}}\|a'\|_{L^1(\mathbb{R})}.$$

$(iii)$ Conclude that there exists $C_0(c_0)$ such that

$$|I(t)| \leq \frac{C_0}{|t|^{\frac{1}{2}}}\|a'\|_{L^1}.$$

$(iv)$ Application : Consider the *Airy equation*

$$\partial_t u + \partial^3_{xxx} u = 0$$

with data $u_0$ integrable and with Fourier transform supported in

$$[-2,-1/2] \cup [1/2, 2].$$

(a) Show that the $L^2$ norm is conserved. Write $u(t) = k_t \star u_0$ for a suitable function $k_t$ and conclude

$$\|u(t)\|_{L^\infty} \leq C|t|^{-\frac{1}{2}}\|u_0\|_{L^1}.$$

Hint: use the fact that if $\varphi$ is smooth with support in $\{\frac{1}{3} \leq |\xi| \leq 3\}$ and equal to 1 on $\{\frac{1}{2} \leq |\xi| \leq 2\}$, then $\widehat{u}_0 = \varphi\widehat{u}_0$.

(b) What kind of $L^p$–$L^{p'}$ estimate do we obtain if $\widehat{u}_0$ is supported in the set $[-2\lambda, -\lambda/2] \cup [\lambda/2, 2\lambda]$ ?

**Exercice 5.4** (A symmetry of the harmonic oscillator)**.** We consider the cubic non linear harmonic oscillator in dimension 2

$$i\partial_t u + \Delta u - |x|^2 u + u|u|^2 = 0.$$

$(i)$ We define the renormalization

$$\left|\begin{array}{l} u(t,x) = \frac{1}{L}w(s,y) \\ w(s,y) = e^{-i\frac{b(s)|y|^2}{4}}v(s,y) \\ y = \frac{x}{L},\quad \frac{ds}{dt} = \frac{1}{L^2}. \end{array}\right.$$

Show that $v(s,y)$ satisfies the same equation iff

$$\left|\begin{array}{l} \frac{L_s}{L} + b = 0 \\ \frac{b_s}{4} - \left(\frac{b^2}{4} + L^4\right) - \frac{b}{2}\frac{L_s}{L} = -1 \end{array}\right. \tag{5.29}$$

where $f_s = \frac{df}{ds}$.

$(ii)$ Integrate the dynamical system (5.29) in time $t$.
Hint: look for a conserved quantity and draw the phase portrait.

# Chapter 6

# Scattering and blow up for (NLS)

The aim of this chapter is to first solve locally in time the Cauchy problem associated to the nonlinear Schrödinger equation

$$\left\{ \begin{array}{l} i\partial_t u + \Delta u + \varepsilon u|u|^{p-1} = 0, \quad (t,x) \in [0,T[\times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{array} \right. \tag{6.1}$$

where $p > 1$ and $\varepsilon \in \{-1, 1\}$ dictates the nature of the nonlinearity: focusing for $\varepsilon = 1$, defocusing for $\varepsilon = -1$.

The nature of the problem is similar to the Cauchy-Lipshitz Theorem for ode: is the knowledge of $u_0$ sufficient to ensure the existence and uniqueness of a maximal in time solution? Then is this solution global, or on the contrary can it blow up in finite time? The approach is indeed to reformulate (6.1) as an ode in a Banach space, the heart of the matter being the choice of the Banach space.

In this chapter, we shall see how Strichartz estimates provide a well adapted functional setting to solve the Cauchy problem. We will then derive classical fundamental results concerning scattering in the defocusing case, and blow up in the focusing case.

## 6.1 The local Cauchy problem

The main result of this chapter is the resolution of (6.1) seen as an ode in the Sobolev space $H^1$. The choice of the $H^1$ space is dictated by the conservation laws as we shall see.

**Theoreme 6.1.1** (Local Cauchy problem in the energy space)**.** *Let $d \geq 1$ et $u_0 \in H^1(\mathbb{R}^d)$. Assume*

$$1 < p < \left\{ \begin{array}{ll} +\infty & si \;\; d = 1, 2 \\ \frac{d+2}{d-2} & si \;\; d \geq 3. \end{array} \right. \tag{6.2}$$

*Then there exists a maximal time $T(u_0) > 0$ such that* (6.1) *admits a unique maximal solution $u \in \mathcal{C}([0,T[; H^1)$. Moreover, there exist two universal constants $C, \alpha > 0$ independent of $u_0$ such that*

$$T(u_0) \geq C\|u_0\|_{H^1}^{-\alpha}.$$

*Finally, there holds the blow up criterion:*

$$T < +\infty \Rightarrow \lim_{t \nearrow T} \|u(t)\|_{H^1} = +\infty. \tag{6.3}$$

In other words, for not too strong nonlinearities (6.2), the Cauchy is well posed and *sub-critical* in the energy space in the sense of (6.3). The local in time theory does not depend on the nature (focusing or defocusing) of the non linearity.

The proof of the general Theorem 6.1.1 is given in [7]. In order to simplify the exposition and extract the essence of the argument, we restrict the presentation to the model[1] case [2] $d=2$ et $p=3$,, see also Exercice 6.1 for a simplified proof for $d=1$.

### 6.1.1 Picard contraction

The structure of the proof, due to Ginibre et Velo [16], is remarkably simple and robust: a Picard fixed point in a suitable Banach space. Indeed, through Duhamel formula, the heart of the proof is to exhibit a fixed point of the map

$$\Phi(u)(t,x) = S(t)u_0(x) + i\varepsilon \int_0^t S(t-s)\left(u(s,x)|u(s,x)|^2\right)\, ds. \tag{6.4}$$

The difficulty is to exhibit a function space where $\Phi$ is a contraction. For example in dimension $d=2$, $H^1$ is not an algebra and hence $u(t)\in H^1$ *does not* ensure a priori that $u(t)|u(t)|^2\in H^1$. Strichartz estimates as studied in chapter 5, will provide the needed gain of integrability with respect to Sobolev estimates which will allow us to close the control of the nonlinear term.
To ease the presentation, we note

$$\|u\|_{L^p_T L^q_x} \stackrel{\text{def}}{=} \left(\int_0^T \|u(t,\cdot)\|^p_{L^q}\, dt\right)^{\frac{1}{p}} \quad \text{for } T>0.$$

More generally, for $E$ Banach space, we let

$$\|u\|_{L^p_T E} \stackrel{\text{def}}{=} \left(\int_0^T \|u(t)\|^p_E\, dt\right)^{\frac{1}{p}}.$$

Hölder's inequality implies:

$$\left\|\prod_{j=1}^r u_j\right\|_{L^p_T L^q_x} \le \prod_{j=1}^r \|u_j\|_{L^{p_j}_T L^{q_j}_x} \tag{6.5}$$

with

$$\frac{1}{p} = \sum_{j=1}^r \frac{1}{p_j}, \quad \frac{1}{q} = \sum_{j=1}^r \frac{1}{q_j}, \quad 1\le p,q,p_j,q_j\le +\infty.$$

In dimension $d=2$, the pairs

$$(\infty, 2) \quad \text{et} \quad (3,6)$$

are admissible, and the space time Lebesgue norm

$$\|u\|_{S_T} = \max\{\|u\|_{L^\infty_T L^2_x}, \|u\|_{L^3_T L^6_x}\}, \tag{6.6}$$

will play a fundamental role. More precisely, we introduce the space time Banach space:

$$X_T = \{u:\ \ \|u\|_{X_T} = \|u\|_{S_T} + \|\nabla u\|_{S_T} < +\infty\} \quad \text{avec } \nabla \stackrel{\text{def}}{=} \nabla_x = (\partial_{x_1},\cdots,\partial_{x_d}).$$

The key to the proof of Theorem 6.1.1 is:

[1] and physically relevant
[2] et physiquement pertinent

**Proposition 6.1.1** ($\Phi$ is a contraction in small time)**.** *There eixst universal constants $C_1, C_2 > 1$ such that for all $u_0 \in H^1$, if*

$$0 < T < \frac{C_1}{\|u_0\|_{H^1}^3} \quad \textit{and} \quad \overline{B}_T = \{u \in X_T : \quad \|u\|_{X_T} \leq C_2 \|u_0\|_{H^1}\}, \tag{6.7}$$

*then $\Phi : \overline{B}_T \to \overline{B}_T$ is a strict contraction.*

*Proof of Proposition 6.1.1.* Let us prove that $\Phi$ is a strict contraction on $\overline{B}_T$ for $T$ small enough:

$$\exists k < 1 \ \text{ t.q. } \ \forall (u,v) \in \overline{B}_T \times \overline{B}_T, \quad \|\Phi(u) - \Phi(v)\|_{X_T} \leq k \|u - v\|_{X_T}.$$

Indeed,

$$\Phi(u)(t) - \Phi(v)(t) = i\varepsilon \int_0^t S(t-s) \left(u(s,x)|u(s,x)|^2 - v(s,x)|v(s,x)|^2\right) ds,$$

and hence inhomogeneous Strichartz estimates and Hölder (6.5) with $(p, p_1, p_2) = (1, 3, 3/2)$ and $(q, q_1, q_2) = (2, 3, 6)$ yield:

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_{S_T} &\lesssim \|u|u|^2 - v|v|^2\|_{L^1_T L^2_x} \lesssim \|(u-v)(|u|^2 + |v|^2)\|_{L^1_T L^2_x} \\ &\lesssim \|u - v\|_{L^3_T L^6_x} \left(\|u\|^2_{L^3_T L^6_x} + \|v\|^2_{L^3_T L^6_x}\right). \end{aligned}$$

We now take one derivative of $\Phi(u)$. Since $S(t)$ and $\nabla$ commute,

$$\left(\nabla \Phi(u)\right)(t,x) = S(t)(\nabla u_0(x)) + i \int_0^t S(t-s) \left[\nabla (u(s,x)|u(s,x)|^2)\right] ds,$$

and hence inhomogeneous Stricharts estimates and Hölder (6.5) with $(p, p_1, p_2, p_3) = (1, 3, 3, 3)$ et $(q, q_1, q_2, q_3) = (2, 6, 6, 6)$ ensure:

$$\begin{aligned} \|\nabla \Phi(u) - \nabla \Phi(v)\|_{S_T} &\lesssim \|\nabla(u|u|^2) - \nabla(v|v|^2)\|_{L^1_T L^2_x} \\ &\lesssim \||\nabla(u-v)|(|u|^2 + |v|^2)\|_{L^1_T L^2_x} + \||u-v|(|\nabla u| + |\nabla v|)(|u| + |v|)\|_{L^1_T L^2_x} \\ &\lesssim \|\nabla(u-v)\|_{L^3_T L^6_x} \left(\|u\|^2_{L^3_T L^6_x} + \|v\|^2_{L^3_T L^6_x}\right) \\ &\quad + \|u - v\|_{L^3_T L^6_x} \left(\|\nabla u\|_{L^3_T L^6_x} + \|\nabla v\|_{L^3_T L^6_x}\right)\left(\|u\|_{L^3_T L^6_x} + \|v\|_{L^3_T L^6_x}\right). \end{aligned}$$

The fundamental observation is that both estimates are *subcritical*[3], which will allow us to show that $\Phi$ is a contraction in $B_T$ for $T = T(\|u_0\|_{H^1})$ small enough. Indeed, using the Sobolev embedding Theorem for $d = 2$:

$$\|u\|_{L^p} \lesssim \|u\|_{H^1} \quad \forall p \in [2, +\infty[.$$

Hence

$$\|u\|_{L^3_T L^6_x} \lesssim \|u\|_{L^3_T H^1_x} \lesssim T^{\frac{1}{3}} \|u\|_{L^\infty_T H^1_x}, \tag{6.8}$$

and we obtain

$$\|\Phi(u) - \Phi(v)\|_{S_T} \lesssim T^{\frac{2}{3}} \|u - v\|_{L^3_T L^6_x} \left(\|u\|^2_{L^\infty_T H^1_x} + \|u\|^2_{L^\infty_T H^1_x}\right).$$

[3]which in the general case is equivalent to assumption (6.2)

and

$$\|\nabla\Phi(u)-\nabla\Phi(v)\|_{S_T} \lesssim T^{\frac{2}{3}}\Big(\|\nabla(u-v)\|_{L^3_TL^6_x}\left(\|u\|^2_{L^\infty_TH^1_x}+\|u\|^2_{L^\infty_TH^1_x}\right)$$
$$+\|u-v\|_{L^\infty_TH^1_x}\left(\|u\|_{L^\infty_TH^1_x}+\|u\|_{L^\infty_TH^1_x}\right)\left(\|\nabla u\|_{L^3_TL^6_x}+\|\nabla v\|_{L^3_TL^6_x}\right)\Big).$$

Hence *in fine* the estimate for some universal constant $c_1>0$:

$$\forall (u,v)\in X_T\times X_T,\quad \|\Phi(u)-\Phi(v)\|_{X_T}\leq c_1T^{\frac{2}{3}}\left(\|u\|^2_{X_T}+\|v\|^2_{X_T}\right)\|u-v\|_{X_T}. \tag{6.9}$$

It remains to prove that $\Phi$ sends $\overline{B}_T$ into $\overline{B}_T$ for $T$ small enough. Indeed, we apply (6.9) with $u$, and $v\equiv 0$. The homogeneous Strichartz estimate ensures

$$\|\Phi(0)\|_{X_T}=\|S(t)u_0\|_{X_T}\lesssim \|u_0\|_{H^1},$$

and hence there exists $c_2>0$ universal such that

$$\forall u\in X_T,\quad \|\Phi(u)\|_{X_T}\leq c_2\|u_0\|_{H^1}+c_2T^{\frac{2}{3}}\|u\|^3_{X_T}.$$

Choose

$$C_2=2c_2$$

in (6.7) and let $u_0\in\overline{B}_T$, then

$$\|\Phi(u)\|_{X_T}\leq c_2\left(\|u_0\|_{H^1}+8c_2^3T^{\frac{2}{3}}\|u_0\|^3_{H^1}\right)\leq 2c_2\|u_0\|_{H^1}$$

as soon as

$$8c_2^3T^{\frac{2}{3}}\|u_0\|^2_{H^1}\leq 1\quad \text{i.e.}\quad T\leq\left(\frac{1}{8c_2^3\|u_0\|^2_{H^1}}\right)^{\frac{3}{2}}. \tag{6.10}$$

For such a time $T$, the closed ball $\overline{B}_T$ is stable by $\Phi$, and hence using (6.9), $\Phi$ is Lipschitz on $\overline{B}_T$ with modulus

$$k\leq 2c_1T^{\frac{2}{3}}C_2^2\|u_0\|^2_{H^1}<1\ \text{ for }\ T<\left(\frac{1}{2c_1C_2^2\|u_0\|^2_{H^1}}\right)^{\frac{3}{2}}.$$

This concludes the proof of Proposition 6.1.1. □

### 6.1.2 Proof of Theorem 6.1.1

We may now conclude the proof of Theorem 6.1.1.

**step 1** Existence and regularity of a solution. Let $u_0\in H^1$, $C_1,C_2$ given by Proposition 6.1.1 and

$$T=\frac{C_1}{2\|u_0\|^3_{H^1}},$$

then Picard's Theorem in the metric space $(\overline{B}_T,\|\cdot\|_{X_T})$ ensures that $\Phi$ admits a unique fixed point $u\in\overline{B}_T$. We claim

$$u\in\mathcal{C}([0,T];H^1). \tag{6.11}$$

Indeed, let $v\in\mathcal{C}([0,T];H^1)$. Using the Fourier representation of the semi group, we have:

$$S(t)v\in\mathcal{C}([0,T];H^1),$$

and hence using the Duhamel formula

$$u = \Phi(u) = S(t)\left[u_0 + i\varepsilon\Phi_1(u)(t)\right] \quad \text{with} \quad \Phi_1(u)(t) = \int_0^t S(-s)(u(s)|u(s)|^2)\, ds.$$

Hence we need only check:

$$u \in X_T \quad \text{implies} \quad \Phi_1(u) \in \mathcal{C}([0,T];H^1). \tag{6.12}$$

But $S(t)$ is isometric on $L^2$, and hence using the continuous embedding $H^1 \subset L^6$ and Hölder:

$$\begin{aligned} \|\Phi_1(u)(t') - \Phi_1(u)(t)\|_{L^2_x} &= \Big\|\int_t^{t'} S(-s)(u(s)|u(s)|^2)\, ds\Big\|_{L^2_x} \leq \int_t^{t'} \|u|u|^2(s)\|_{L^2_x}\, ds \\ &\lesssim |t-t'|\|u\|^3_{L^\infty_T H^1_x} \lesssim |t-t'|\|u\|^3_{X_T}, \end{aligned}$$

and similarily after applying one derivative in space:

$$\begin{aligned} \|\nabla\Phi_1(u)(t') - \nabla\Phi_1(u)(t))\|_{L^2_x} &\lesssim \int_t^{t'} \|\nabla(u|u|^2)(s)\|_{L^2_x}\, ds \lesssim \int_t^{t'} \|\nabla u(s)\|_{L^6_x}\|u(s)\|^2_{L^6_x}\, ds \\ &\lesssim |t-t'|^{\frac{2}{3}}\|u\|^2_{L^\infty_T H^1_x}\|\nabla u\|_{L^3_T L^6_x} \lesssim |t-t'|^{\frac{2}{3}}\|u\|^3_{X_T}, \end{aligned}$$

which concludes the proof of (6.12), and hence (6.11).

**step 2** Uniqueness and blow up criterion. Let $u$ be given by step 1 which is a solution $u \in \mathcal{C}([0,T];H^1)$ of (6.1). Let $v \in \mathcal{C}([0,T];H^1)$ another solution. Let $M$ denote a shared bound for $\|u\|_{L^\infty_T H^1_x}$ and $\|v\|_{L^\infty_T H^1_x}$. The assumptions ensure that $v|v|^2 \in L^1_T L^2_x$ by Sobolev and Hölder, and hence Proposition 5.1.2 ensures

$$v = \Phi(v).$$

But $v \in L^\infty_T H^1_x \subset L^3_T L^6_x$ by Sobolev. Hence by (6.8) and (6.8), for all $T_0 \in ]0,T]$,

$$\begin{aligned} \|u-v\|_{L^3_{T_0}L^6_x} &= \|\Phi(u) - \Phi(v)\|_{L^3_{T_0}L^6_x} \lesssim \|u-v\|_{L^3_{T_0}L^6_x}\left(\|u\|^2_{L^3_{T_0}L^6_x} + \|v\|^2_{L^3_{T_0}L^6_x}\right) \\ &\lesssim T_0^{\frac{2}{3}}\|u-v\|_{L^3_{T_0}L^6_x}\left(\|u\|^2_{L^\infty_{T_0}H^1_x} + \|v\|^2_{L^\infty_{T_0}H^1_x}\right) \lesssim T_0^{\frac{2}{3}} M^2\|u-v\|_{L^3_{T_0}L^6_x}. \end{aligned}$$

Hence there exists $c > 0$ such that

$$\|u-v\|_{L^3_{T_0}L^6_x} \leq \frac{1}{2}\|u-v\|_{L^3_{T_0}L^6_x} \quad \text{avec} \quad T_0 \stackrel{\text{def}}{=} \min\left(\frac{c}{M^3}, T\right).$$

which implies $u = v$ on $[0,T_0]$. Since $T_0$ depends only the $H^1$ bound a priori bound of the solution on $[0,T]$, we may iterate the argument starting at $T_0$, and obtain uniqueness on $[T_0, 2T_0]$, and so on, which yields uniqueness on $[0,T]$.
It remains to prove the blow up criterion (6.3). Let $u \in \mathcal{C}([0,T[;H^1)$ be a maximal solution with $T < +\infty$. Assume by contradiction that there exists $M \geq 0$ finite such that

$$\forall t \in [0,T[,\ \|u(t)\|_{H^1} \leq M. \tag{6.13}$$

Then by (6.7), for all $t_0 \in [0,T[$, we may construct a solution to (6.1) with data $u(t_0)$ at $t = t_0$ on a time interval of length $CM^{-3}$ with $C$ universal (see(6.10)). For $t_0$ such that $T - t_0 < CM^{-3}$, we obtain a new solution defined beyond $T$ which in light of the uniqueness result coincides with $u$ on $[0,T[$, and this contradicts the maximality of $T$.

## 6.2 Conservation laws and global existence

In this section, we aim at understanding under which conditions the local in time solutions of the Cauchy problem provided by Theorem 6.1.1 exist for all times. Here the nature of the singularity, focusing or defocusing, and the algebraic sturcture $u|u|^{p-1}$ of the non linearity, will both play a fundamental role.

### 6.2.1 Symmetries and conservation laws

We describe in this section two fundamental structural facts: the existence of symmetries and the existence of conservation laws, both being connected.

**Proposition 6.2.1** (Symmetries of(NLS)). *Let $u \in \mathcal{C}([0,T[;H^1)$ satisfiying*

$$i\partial_t u + \Delta u + \varepsilon u|u|^{p-1} = 0. \tag{6.14}$$

*Then the following functions are also solutions to* (6.14) *:*

- *Scaling:* $(t,x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$, $\lambda > 0$*;*
- *Translation :* $(t,x) \mapsto u(t, x + x_0)$, $x_0 \in \mathbb{R}^d$*;*
- *Phase :* $(t,x) \mapsto u(t,x)e^{i\gamma}$, $\gamma \in \mathbb{R}$*;*
- *Galilean drift :* $(t,x) \mapsto u(t, x - 2\beta t)e^{i\beta\cdot(x-\beta t)}$, $\beta \in \mathbb{R}$.

Scale invariance plays a fundamental role in the classification of the (NLS) problems through the computation of the scaling parameter.

**Definition 6.2.1** (Scaling parameter). *The* scaling *associated to* (6.1) *is the unique exponent $s_c$ such that the dilation $u(t,x) \rightsquigarrow u_\lambda(t,x) \overset{def}{=} \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ leaves the homogeneous Sobolev norm $\dot{H}^{s_c}$ invariant:*

$$\|u_\lambda(t,\cdot)\|_{\dot{H}^{s_c}} = \|u(\lambda^2 t,\cdot)\|_{\dot{H}^{s_c}}.$$

*Explicitely*[4]*,*

$$s_c = \frac{d}{2} - \frac{2}{p-1}. \tag{6.15}$$

*We say that* (6.1) *is $\dot{H}^{s_c}$ critical.*

*Example.* Let $p = 3$. Then $d = 2$ is $s_c = 0$, problem is $L^2$*–critical* Then $d = 1$ is $s_c = -\frac{1}{2}$, the equation is $L^2$*–sub critical* since the critical space $\dot{H}^{-\frac{1}{2}}$ is below $L^2$ in the Sobolev ladder. Finally $d = 3$ is $s_c = \frac{1}{2}$, the equation is $\dot{H}^{\frac{1}{2}}$ *critical*, and $L^2$–super critical. These three cases correspond to three relevant physical situations with dramatic changes in the behaviour of solutions (see Theorem 6.2.1)

Noether's theorem ensures that symmetries imply conservation laws (see e.g. [22] ou [39]).

**Proposition 6.2.2** (Conservation lawe). *Let $u_0 \in H^1$ and $u \in \mathcal{C}([0,T[;H^1)$ the solution to* (6.1)*. Then for all $t \in [0,T[$ :*

*(i) Conservation of mass:*

$$\int_{\mathbb{R}^d} |u(t,x)|^2\,dx = \int_{\mathbb{R}^d} |u_0(x)|^2\,dx. \tag{6.16}$$

[4] elementary in Fourier

*(ii) Conservation of energy:*

$$E(u(t)) \stackrel{def}{=} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,dx - \frac{\varepsilon}{p+1}\int_{\mathbb{R}^d} |u(t,x)|^{p+1}\,dx = E(u_0). \tag{6.17}$$

*(iii) Conservation of kinetic momentum*[5]*:*

$$M(u(t)) \stackrel{def}{=} \mathrm{Im}\left(\int_{\mathbb{R}^d} \nabla u(t,x)\overline{u(t,x)}\,dx\right) = M(u_0). \tag{6.18}$$

The physical interpretation of these conservation laws are clear: conservation of the total probability of presence for the mass, and the total kinetic momemtum for the moment. The energy $E(u)$ is the sum of the kinetic energy and the potential energy. For $\varepsilon = 1$, the minus sign in the potential energy indicates a *focusing* nonlinearity which acts against the natural spreading of the wave packet.

*Proof of Proposition 6.2.2.* The proof relies on a formal computation where all integrals are defined on $\mathbb{R}^d$, and where one uses the integration by parts formula with vanishing boundary term at $|x| \to +\infty$:

$$\int \nabla u \cdot \overline{\nabla v}\,dx = -\int \Delta u\,\overline{v}\,dx.$$

Let us assume to begin with that $u$ is space time smooth and decaying at $+\infty$ in space, as well as all its derivatives. Then for the mass:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int |u(t,x)|^2\,dx\right\} = \mathrm{Re}\left(\int \partial_t u(t,x)\overline{u(t,x)}\,dx\right) = \mathrm{Im}\left(\int i\partial_t u(t,x)\overline{u(t,x)}\,dx\right)\\
= &\; -\mathrm{Im}\left(\int (\Delta u + \varepsilon u|u|^{p-1})(t,x)\,\overline{u(t,x)}dx\right) = \mathrm{Im}\left(\int |\nabla u(t,x)|^2\,dx\right) = 0.
\end{aligned}$$

For the energy

$$\begin{aligned}
\frac{d}{dt}E(u) &= \mathrm{Re}\left(\int \nabla\partial_t u \cdot \overline{\nabla u}\,dx - \varepsilon\int \partial_t u\overline{u|u|^{p-1}}\,dx\right)\\
&= -\mathrm{Re}\left(\int \partial_t u\left[\overline{\Delta u + \varepsilon u|u|^{p-1}}\right]\,dx\right) = -\mathrm{Im}\left(\int i\partial_t u\left[\overline{\Delta u + \varepsilon u|u|^{p-1}}\right]\,dx\right)\\
&= \mathrm{Im}\left(\int |\Delta u + \varepsilon u|u|^{p-1}|^2\,dx\right) = 0.
\end{aligned}$$

For the momentum, let $j \in \{1,\cdots,d\}$,

$$\begin{aligned}
\frac{d}{dt}M(u) &= \mathrm{Im}\left(\int \partial^2_{tj}u\,\overline{u}\,dx + \int \partial_j u\overline{\partial_t u}\,dx\right)\\
&= -2\,\mathrm{Im}\left(\int \partial_t u\overline{\partial_j u}\,dx\right) = 2\,\mathrm{Re}\left(\int i\partial_t u\overline{\partial_j u}\,dx\right)\\
&= -2\,\mathrm{Re}\left(\int (\Delta u + \varepsilon u|u|^{p-1})\,\overline{\partial_j u}\,dx\right) = 0
\end{aligned}$$

[5] $M(u)$ is a vector with components $\mathrm{Im}\left(\int_{\mathbb{R}^d}\partial_j u(t,x)\overline{u(t,x)}\,dx\right)$, $1 \le j \le d$.

where we used the integration by parts formula for functions null at $+\infty$:

$$\mathrm{Re}\left(\int \Delta u\,\overline{\partial_j u}\,dx\right) = -\,\mathrm{Re}\left(\sum_{k\neq j}\int \partial_k u\,\overline{\partial^2_{jk}u}\,dx\right) = 0.$$

These three computations can be justified for $u \in \mathcal{C}([0,T[;H^1)$ modulo a regularization argument, we refer to [7] for a complete exposition of the argument. □

**Remark 6.2.1.** *We may now reinterpret the constraint* (6.2) *on the size of* $p$. *Let* $s_c$ *be the scaling parameter associated to*(6.1)*:*

$$s_c = \frac{d}{2} - \frac{2}{p-1}\cdot$$

*Then* (6.2) *is equivalent to*

$$s_c < 1,$$

*ie* (6.1) *is* $H^1$*–sub-critical. Equivalently,* (6.2) *is*

$$p+1 < \frac{2d}{d-2} = 2^* \quad \textit{where} \quad \dot{H}^1 \hookrightarrow L^{2^*} \quad \textit{for} \quad d \geq 3,$$

*and hence the Sobolev embedding Theorem ensures that* $E(u)$ *given by* (6.17) *is finite for* $u \in H^1$. *So are the other two conservation laws, and hence* $H^1$ *is the minimum regularity for which the three conservation laws of* (6.1) *are well defined, hence the relevance of a Cauchy theory in this* energy *space*[6].

### 6.2.2 Global existence

We are now in position to state the fundamental global existence Theorem.

**Theoreme 6.2.1** (Global existence)**.** *Let* $d \geq 1$ *and* $p > 1$ *satisfiying* (6.2). *Let* $s_c$ *be the scaling exponent given by* (6.15). *Assume one of the following two cases:*

*(i) Defocusing energy subcritical non linearity:* $\varepsilon = -1$ *and* $s_c < 1$*;*

*(ii) Focusing mass subcritical non linearity:* $\varepsilon = 1$ *and* $s_c < 0$.

*Then for all* $u_0 \in H^1$, *the solution to the Cauchy problem* (6.1) *given by Theorem 6.1.1 is global and bounded in* $H^1$*:*

$$T = +\infty \quad \textit{and} \quad \sup_{t\in\mathbb{R}^+} \|u(t)\|_{H^1} \leq C(u_0)$$

*where* $C(u_0)$ *depends only on the initial data.*

*Proof of Theorem 6.2.1.* Let $u_0 \in H^1$ and $u \in \mathcal{C}([0,T[;H^1)$ be the maximal solution to (6.1) given by Theorem 6.1.1. Global existence follows from an a priori bound on $\|u(t)\|_{H^1}$ which coupled to the blow up criterion (6.3) implies $T = +\infty$. The uniform control of the $H^1$ norm is obvious in the defocusing case $\varepsilon = -1$ since the mass is conserved and both terms in the

[6] But the Cauchy problem may be perfectly well posed in other spaces. Typically (6.1) for $p = 3$ and $d = 2$ has a well posed Cauchy problem in $L^2$, see Exercice 6.2.

energy (6.17) are under control. In the focusing case $\varepsilon = 1$, the Gagliardo-Nirenberg inequality of chapter 4 yields:

$$\|u\|_{L^{p+1}} \leq C\|u\|_{L^2}^{1-\sigma}\|\nabla u\|_{L^2}^{\sigma} \quad \text{with} \quad -\sigma + \frac{d}{2} = \frac{d}{p+1}\cdot$$

We inject this estimate into the conservation of the energy and lower bound:

$$\begin{aligned} E(u_0) &= E(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - C\|u\|_{L^2}^{(p+1)(1-\sigma)}\|\nabla u\|_{L^2}^{(p+1)\sigma} \\ &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - C\|u_0\|_{L^2}^{(p+1)(1-\sigma)}\|\nabla u\|_{L^2}^{(p+1)\sigma} \end{aligned} \tag{6.19}$$

where use the conservation of the $L^2$ norm in the last step. We now observe

$$s_c < 0 \leftrightarrow p < 1 + \frac{4}{d} \Leftrightarrow (p+1)\sigma = \frac{d(p-1)}{2} < 2.$$

Hence the function

$$x \to \frac{1}{2}x^2 - C\|u_0\|_{L^2}^{(p+1)(1-\sigma)}x^{(p+1)\sigma}$$

diverges to $+\infty$ as $x \to +\infty$, and (6.19) implies

$$\sup_{t\in[0,T[} \|\nabla u(t)\|_{L^2} \leq C(u_0),$$

which using $L^2$ conservation yields the a priori bound on the $H^1$ norm. $\square$

## 6.3 Scattering and blow up

In this section, we give a further qualitative description of the flow in the continuation of Theorem 6.2.1. We will show that global existence for the energy subcritical defocusing NLS is in fact scaterring, and that for the focusing problem, the global existence criterion $s_c < 0$ is sharp.

Let us stress that there is no abstract general route map for the study of non linear problems. Most known results rely on *mononotonicity formulas*[7].

### 6.3.1 The virial identity

A fundamental monotonicity formula for (6.1) relies on the *virial identity* which makes sense in the virial space $\Sigma$ defined in (5.22).

**Lemma** (Virial identity). *Let $u_0 \in \Sigma$ and $u \in \mathcal{C}([0,T[;\Sigma)$ the corresponding solution to* (6.1) *given by Theorem 5.3.1. Then*

$$\frac{d}{dt}\int |x|^2|u(t,x)|^2\,dx = 4\,\mathrm{Im}\left(\int x\cdot\nabla u\,\overline{u}\,dx\right) \tag{6.20}$$

$$\frac{1}{2}\frac{d}{dt}\,\mathrm{Im}\left(\int x\cdot\nabla u\,\overline{u}\,dx\right) = \int |\nabla u|^2\,dx - \varepsilon\left[\frac{d}{2} - \frac{d}{p+1}\right]\int |u|^{p+1}\,dx. \tag{6.21}$$

[7] like for Perelman's proof of the Poincaré conjecture which heart is a monotonicity statement for the Ricci flow of surfaces.

*Proof of Lemma 6.2.1.* The proof follows by direct computation.

**step 1** Pohozaev identity. We will need the celebrated Pohozaev indentity:

$$\int \Delta u \left(\frac{d}{2}u + x\cdot\nabla u\right) dx = -\int |\nabla u|^2\, dx. \tag{6.22}$$

By density, we need only prove (6.22) for $u \in D(\mathbb{R}^d)$. Let

$$u_\lambda(x) \stackrel{\text{def}}{=} \lambda^{\frac{d}{2}} u(\lambda x),$$

then

$$\int |\nabla u_\lambda|^2\, dx = \lambda^2 \int |\nabla u|^2\, dx.$$

Deriving this identity with respect to $\lambda$ and evaluating the result at $\lambda = 1$ yields:

$$\int \nabla u \cdot \nabla\overline{\left(\frac{d}{2}u + x\cdot\nabla u\right)}\, dx = \int |\nabla u|^2\, dx.$$

Integrating by parts the left hand side yields (6.22). Observe that the same argument yields

$$\int |u_\lambda|^q\, dx = \lambda^{\frac{dq}{2}-d} \int |u|^q\, dx, \qquad q \geq 2,$$

and hence

$$\mathrm{Re}\int u\left(\frac{d}{2}\overline{u} + x\cdot\nabla\overline{u}\right)|u|^{q-2}\, dx = \left(\frac{d}{2} - \frac{d}{q}\right)\|u\|_{L^q}^q. \tag{6.23}$$

**step 2** Virial. Assuming $u$ is space time and well decaying, we compute:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |x|^2|u(t,x)|^2\, dx &= \mathrm{Re}\left(\int |x|^2\partial_t u\,\overline{u}\, dx\right) = \mathrm{Im}\left(\int |x|^2 i\partial_t u\,\overline{u}\, dx\right)\\
&= -\mathrm{Im}\left(\int |x|^2(\Delta u + \varepsilon u|u|^{p-1})\overline{u}\, dx\right) = \mathrm{Im}\left(\int \nabla u\cdot(|x|^2\overline{\nabla u} + 2x\overline{u})\, dx\right)\\
&= 2\,\mathrm{Im}\left(\int x\cdot\nabla u\overline{u}\, dx\right)
\end{aligned}$$

and (6.20) is proved. Then:

$$\begin{aligned}
\frac{d}{dt}\mathrm{Im}\left(\int x\cdot\nabla u\,\overline{u}\, dx\right) &= \mathrm{Im}\left(\int x\cdot\nabla\partial_t u\,\overline{u} + x\cdot\nabla u\,\overline{\partial_t u}\, dx\right) \qquad (6.24)\\
= -\mathrm{Im}\left(\int \partial_t u\left[\nabla\cdot(x\overline{u}) + \overline{x\cdot\nabla u}\right] dx\right) &= -2\,\mathrm{Im}\left(\int \partial_t u\left[\overline{\frac{d}{2}u + x\cdot\nabla u}\right] dx\right)\\
= 2\,\mathrm{Re}\left(\int i\partial_t u\left[\overline{\frac{d}{2}u + x\cdot\nabla u}\right] dx\right) &= -2\,\mathrm{Re}\left(\int \left[\Delta u + \varepsilon u|u|^{p-1}\right]\left[\overline{\frac{d}{2}u + x\cdot\nabla u}\right] dx\right).
\end{aligned}$$

We now use Pohozaev (6.22):

$$\begin{aligned}
-2\,\mathrm{Re}\left(\int \Delta u\left[\overline{\frac{d}{2}u + x\cdot\nabla u}\right] dx\right) &= 2\,\mathrm{Re}\left(\int \nabla u\cdot\nabla\left[\overline{\frac{d}{2}u + x\cdot\nabla u}\right] dx\right)\\
&= 2\int |\nabla u|^2\, dx.
\end{aligned}$$

and the non linear is computed from (6.23):

$$-2\,\mathrm{Re}\left(\int \varepsilon u|u|^{p-1}\overline{\left[\frac{d}{2}u + x\cdot\nabla u\right]}\right)dx = -2\varepsilon\left[\frac{d}{2} - \frac{d}{p+1}\right]\int |u|^{p+1}dx,$$

and (6.21) is proved. The proof for $u \in \mathcal{C}([0,T[;\Sigma)$ relies on classical but lenghty regularization arguments, see e.g. [7]. □

There are two spectacular consequences of the virial identity which may seem addressing completely different issues: finite time blow up for $s_c > 0$ in the focusing case, scattering for the energy sub critical defocusing (NLS).

### 6.3.2 Blow up for focusing (NLS)

In this section, we prove the celebrated blow up by virial for (6.1) which appeared in the Russian litterature in the 1950's.

**Theorem** (Finite time blow up)**.** *Let $s_c \geq 0$ for the focusing* (6.1) $\varepsilon = 1$*. Let*

$$u_0 \in \Sigma \quad \textit{with} \quad E_0 < 0. \tag{6.25}$$

*Then the corresponding solution $u \in \mathcal{C}([0,T[;\Sigma)$ to* (6.1) *blows up in finite time.*

**Remark 6.3.1.** *The assumptions of the Theorem are not empty. Let $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $u_0 = a\phi$, $a > 0$, then $u_0 \in \Sigma$ and*

$$E(u_0) = \frac{a^2}{2}\int |\nabla\phi|^2 - \frac{a^{p+1}}{p+1}\int |\phi|^{p+1} < 0$$

*for all $a > a(\phi)$ large enough.*

**Remark 6.3.2.** *This result shows that the global existence criterion $s_c < 0$ of Theorem 6.2.1 is* sharp. The model problem is the cubic two dimensional problem

$$i\partial_t u + \Delta u + u|u|^2 = 0$$

which has been introduced in the 1950's to model the focusing of a laser beam, and is the limiting case $s_c = 0$.

*Proof of Theorem 6.3.2.* Combining the virial identities (6.20), (6.21) with the conservation of energy and the observation $2(p-1)s_c = (p-1)d - 4 \geq 0$, we obtain

$$\begin{aligned} \frac{1}{16}\frac{d^2}{dt^2}\int |x|^2|u|^2\,dx &= \frac{1}{2}\int |\nabla u|^2 - \frac{1}{2}\left[\frac{d}{2} - \frac{d}{p+1}\right]\int |u|^{p+1}\,dx \\ = \quad E(u) - \frac{1}{2}\left[\frac{d(p-1)}{2(p+1)} - \frac{2}{p+1}\right]\int |u|^{p+1}\,dx &= E_0 - \frac{(p-1)s_c}{2(p+1)}\int |u|^{p+1}\,dx \leq E_0. \end{aligned}$$

The positive quantity $\int |x|^2|u(t,x)|^2dx$ therefore lies below an inverted parabola with dominant coefficient $E_0 < 0$, and hence it must become non positive in finite time. Hence the solution cannot exist for all times. □

Let us stress that this kind of blow up result is very rare, and such questions for non linear PDE's are mostly open. The analogous problem for the Navier-Stokes equation describing the evolution of a three dimensionl incompressible fluid is one of the Clay Millenium problems.

For (6.1), the blow up virial argument is spectacular by its simplicity and its robustness. It not only proves that blow up happens, it exhibits an *open* region of phase space where blow up occurs, ie $E_0 < 0$. The virial algebra which was discovered in the study of nonlinear optics (see [41]) is more universal than one could think, and was for example used to prove the formation of shock in compressible fluid dynamics, [35]. The argument is however *unstable* by perturbation of the equation, and gives no insight into the nature of the singularity. The aim of this section was to give a glimpse at this class of problems which is the subject of an intense research activity since the beginning of the new Millenium.

### 6.3.3 Scattering for defocusing (NLS)

We conclude this chapter by completing Theorem 6.2.1 in the defocusing case, and proving that asympotically in time, solutions scatter ie behave like linear waves.

**Proposition 6.3.1** (Scattering in $\Sigma$)**.** *Let $u_0 \in \Sigma$ and $u \in \mathcal{C}(\mathbb{R};\Sigma)$ be the global solution to (6.1) given by Theorem 6.2.1 in the defocusing case $\varepsilon = -1$ with $0 < s_c < 1$. Then $\exists u_{+\infty} \in \Sigma$ such that*

$$\lim_{t\to+\infty} \|u(t,\cdot) - S(t)u_{+\infty}\|_{L^2} = 0. \tag{6.26}$$

**Remark 6.3.3.** *Proposition 6.3.1 does not cover the values $s_c < 0$ of Theorem 6.2.1. The result remains true for $p^* < p \leq 1+\frac{4}{d}$, but the proof is more complicated, and counterexamples exist for $p < p^*$ where* modified *scattering is expected.*

*Proof of Proposition 6.3.1.* For the sake of simplicity, we restrict the analysis ro $d \geq 3$. The route map is similar for $d = 1, 2$.

**step 1** Pointwise decay. Let

$$F(t) \stackrel{\text{def}}{=} \int |xu + 2it\nabla u|^2 dx + \frac{8t^2}{p+1}\int |u|^{p+1}dx.$$

Then

$$F(t) = \int |x|^2|u|^2 dx - 4t\,\mathrm{Im}\left(\int x\cdot\nabla u\,\overline{u}\,dx\right) + 8t^2E(u_0)$$

and hence using the virial identities (6.20), (6.21) with $\varepsilon = -1$:

$$\frac{dF}{dt} = -\frac{4t}{p+1}[d(p-1)-4]\int |u|^{p+1}\,dx \leq 0. \tag{6.27}$$

Moreover, let $v(t,x) = e^{-i\frac{|x|^2}{4t}}u(t,x)$, then a direct computation reveals

$$F(t) = 8t^2E(v)$$

and hence by the monotonicity (6.27):

$$t^2E(v) \leq F(0) = \|xu_0\|^2_{L^2} \quad \text{i.e.} \quad 4t^2\|\nabla v(t)\|^2_{L^2} \leq \|xu_0\|^2_{L^2}. \tag{6.28}$$

Let

$$2 \le r \le \frac{2d}{d-2}\cdot$$

We now transform the decay estimate (6.28) using the Gagliardo-Nirenberg inequality :

$$\|v\|_{L^r} \le c_{r,d}\|\nabla v\|_{L^2}^{\alpha}\|v\|_{L^2}^{1-\alpha} \quad \text{with} \quad \alpha = \frac{d}{2} - \frac{d}{r}\cdot$$

Hence using the conservation of mass:

$$\|u(t)\|_{L^r} = \|v(t)\|_{L^r} \lesssim C(u_0)\|\nabla v(t)\|_{L^2}^{\frac{d}{2}-\frac{d}{r}} \lesssim \frac{C(u_0)}{t^{\frac{d}{2}-\frac{d}{r}}}$$

which is the non linear analogue of the pointwise decay bound (5.7).

**step 2** Controlling the Strichartz norm. To be continued.

□

## 6.4 Exercices

**Exercice 6.1** (The Cauchy problem (6.1) in dimension $d = 1$)**.**

($i$) Let $p \in \mathbb{N} \setminus \{0, 1\}$, $u_0 \in H^1$.

- (a) Using only the $L^2$ isometry property of $S(t)$ and the Sobolev injection $H^1 \hookrightarrow L^\infty$, show that (6.4) is a contraction on a well chosen ball of the space $X_T = \mathcal{C}([0, T]; H^1)$ equipped with the norm $\|\cdot\|_{L^\infty([0,T];H^1)}\cdot$
- (b) Solve the local Cauchy problem (6.1.1) for $d = 1$.

($ii$) We now assume $p = 3$ (for the sake of simplicity), and we pick $u_0 \in L^2$.

- (a) Show that there exists $T > 0$ such that (6.1) admits a unique solution in

$$Y_T \stackrel{\text{def}}{=} \mathcal{C}([0, T]; L^2(\mathbb{R})) \cap L^4([0, T]; L^\infty(\mathbb{R})).$$

- (b) Using chapter 5, show that the mass is conserved.
- (c) Show that the maximal solution constructing from $u_0$ is global, and belongs to $Y_T$ for all $T > 0$.
- (d) Show that for such a solution $u$, for all $0 \le t_0 \le t$,

$$\|u\|_{L^4([t_0,t];L^\infty)} \le C\|u_0\|_{L^2}\left(1 + \int_{t_0}^{t} \|u\|_{L^\infty}^2 \, d\tau\right),$$

and deduce that there holds for some universal constant $C'$:

$$\|u\|_{L^4([0,t];L^\infty)} \le C'\|u_0\|_{L^2}(1 + t^{\frac{1}{4}}\|u_0\|_{L^2}) \quad \text{for all } t \ge 0.$$

**Exercice 6.2** (The critical Cauchy problem)**.** We consider the Cauchy problem (6.1) with $d = 2$, $p = 3$ and $u_0 \in L^2$ only.

($i$) Show that (6.4) is contraction on a well chosen ball of $S_T = \mathcal{C}([0, T]; L^2) \cap L^3([0, T]; L^6)$ equipped with (6.6).

(*ii*) Enounce a result of existence and uniqueness in $S_T$, and show that the mass is conserved (6.16).

(*iii*) Are all solutions global? Which is the new blow up criterion replacing (6.3)?

(*iv*) We now assume that $\|u_0\|_{L^2}$ is small.

(a) Show that global existence holds in the space $S_\infty \stackrel{\text{def}}{=} \mathcal{C}(\mathbb{R}^+; L^2) \cap L^3(\mathbb{R}^+; L^6)$.

(b) Let $v_\infty \stackrel{\text{def}}{=} i\varepsilon \int_0^{+\infty} S(-\tau)(u(\tau)|u(\tau)|^2)\, d\tau$. Check that $v_\infty \in L^2$ and compute $u(t) - S(t)v_\infty$.

(c) How can we choose $u_\infty \in L^2$ so that $\lim_{t\to+\infty} \|u(t) - S(t)u_\infty\|_{L^2} = 0$ ?

**Exercice 6.3** (Local existence in $H^2$ for (6.1) cubic in dimension 2)**.** We consider

$$(S_\varepsilon) \qquad i\partial_t u + \Delta u + \varepsilon|u|^2 u = 0 \quad \text{dans } \mathbb{R}\times\mathbb{R}^2, \qquad \varepsilon \in \{-1, 1\}$$

with data $u_0 \in H^2(\mathbb{R}^2)$.

Let $E_T = \mathcal{C}([0,T]; H^2)$ and $u_L = S(t)u_0$. For $u \in E_T$, we let

$$\forall t \in [0,T],\ \Phi(u)(t) = u_L(t) + i\varepsilon \int_0^t S(t-\tau)((|u|^2 u(\tau))\, d\tau.$$

(*i*) Using Proposition 4.1.2 and Sobolev injections, show that $H^2(\mathbb{R}^2)$ is stable by product.

(*ii*) Show that $\Phi$ is well defined from $E_T$ into $E_T$, and there exist two constants $C_1, C_2 > 0$ such that for all $u, v$ in $B_{E_T}(u_L, R)$,

$$\|\Phi(u) - u_L\|_{E_T} \le C_1 T(R^3 + \|u_0\|_{H^2}^3) \text{ and } \|\Phi(u) - \Phi(v)\|_{E_T} \le C_2 T(R^2 + \|u_0\|_{H^2}^2)\|u - v\|_{E_T}.$$

(*iii*) Conclude that there exists $c > 0$ and a time $T \ge c/\|u_0\|_{H^2}^2$ such that $\Phi$ has a fixed point in $E_T$.

(*iv*) Conclude that there exists $T^* > 0$ such that $(S_\varepsilon)$ with data $u_0 \in H^2$ has a unique maximal solution $u \in \mathcal{C}([0,T^*[; H^2) \cap \mathcal{C}^1([0,T^*[; L^2)$.

(*v*) In this question, we look for a blow up criterion.

(a) Prove the Gagliardo-Nirenberg estimate :

$$\forall u \in H^2(\mathbb{R}^2),\ \|\partial_1 u\|_{L^4}^2 \le 3\|u\|_{L^\infty}\|\partial_{11}^2 u\|_{L^2}.$$

(b) Prove the *tame estimate* :

$$\forall (u,v) \in H^2 \times H^2,\ \|uv\|_{H^2} \le C_0\big(\|u\|_{L^\infty}\|v\|_{H^2} + \|v\|_{L^\infty}\|u\|_{H^2}\big)$$

for some universal constant $C_0$.

(c) Show that there exists $C > 0$ universal such that for all solution $u \in E_T$ of $(S_\varepsilon)$:

$$\forall t \in [0,T],\ \|u(t)\|_{H^2} \le \|u_0\|_{H^2} + C\int_0^t \|u\|_{L^\infty}^2 \|u\|_{H^2}\, d\tau.$$

(d) Conclude that $T^* < +\infty$ implies $\int_0^{T^*} \|u(t)\|_{L^\infty}^2 \, dt = +\infty$. For $\varepsilon = -1$, does this allow to conclude $T^* = +\infty$?

($vi$) Using Theorem 6.2.1, prove global existence in $H^2$ for $\varepsilon = -1$.

**Exercice 6.4** (Cubic wave equation)**.** We consider the Cauchy problem for the non linear wave equation in dimension $d = 3$:

$$(NLW^\varepsilon) \qquad \begin{cases} \partial_{tt}^2 u - \Delta u + \varepsilon u^3 = 0, & (t,x) \in I \times \mathbb{R}^3, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases}$$

where $I$ is an interval of $\mathbb{R}$ containing 0, $u_0 : \mathbb{R}^3 \to \mathbb{R}$ and $u_1 : \mathbb{R}^3 \to \mathbb{R}$ are the data, and $\varepsilon \in \{-1, 1\}\cdot$ The non linearity is defocusing for $\varepsilon = 1$, and focusing for $\varepsilon = -1$.

We also consider the linear wave equation

$$(W) \qquad \begin{cases} \partial_{tt}^2 u - \Delta u = f, & (t,x) \in I \times \mathbb{R}^3, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases}$$

with $u_0 : \mathbb{R}^3 \to \mathbb{R}, \;\; u_1 : \mathbb{R}^3 \to \mathbb{R}$ and $f : I \times \mathbb{R}^3 \to \mathbb{R}$ given.

($i$) Let $f \equiv 0, \; u_0, u_1 \in \mathcal{S}(\mathbb{R}^3)$, show that the solution $u$ to $(W)$ is given by the formula:

$$u(t) = U^+(t)\gamma^+ + U^-(t)\gamma^-$$

with $\mathcal{F}(U^\pm(t)z)(\xi) = e^{\pm it|\xi|}\mathcal{F}z(\xi)$ and

$$\mathcal{F}\gamma^\pm(\xi) = \frac{1}{2}\bigg(\mathcal{F}u_0(\xi) \pm \frac{1}{i|\xi|}\mathcal{F}u_1(\xi)\bigg)\cdot$$

($ii$) Show that the quantity $\|\nabla_{t,x} u(t)\|_{L^2(\mathbb{R}^3)}^2 \stackrel{\text{def}}{=} \|\nabla_x u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_t u(t)\|_{L^2(\mathbb{R}^3)}^2$ is independent of time.

($iii$) Let $\dot{H}^1(\mathbb{R}^3)$ be the closure of $\mathcal{S}(\mathbb{R}^3)$ for the norm $\|z\|_{\dot{H}^1} \stackrel{\text{def}}{=} \|\nabla z\|_{L^2}$. Explain briefly why this set coincides with $L^6(\mathbb{R}^3)$ functions which gradient is in $L^2(\mathbb{R}^3)$, and then show that for all $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, the linear wave $(W)$ with $f \equiv 0$ has a unique solution $u \in \mathcal{C}(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)) \cap \dot{\mathcal{C}}^1(\mathbb{R}; L^2(\mathbb{R}^3))$ (the second space just means $\partial_t u \in \mathcal{C}(\mathbb{R}; L^2(\mathbb{R}^3))$).

($iv$) In the general case $f \neq 0$, show that smooth decaying at infinity solutions of $(W)$ satisfy:

$$\frac{1}{2}\frac{d}{dt}\|(\partial_t u, \nabla u)\|_{L^2}^2 = \int_{\mathbb{R}^3} f \, \partial_t u \, dx$$

and then for all $t \in \mathbb{R}$,

$$\|\nabla_{t,x} u(t)\|_{L^2} \leq \|(\nabla u_0, u_1)\|_{L^2} + \left| \int_0^t \|f\|_{L^2} \, d\tau \right|$$

with

$$\|(\nabla u_0, u_1)\|_{L^2} \stackrel{\text{def}}{=} \sqrt{\|\nabla u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2}\,.$$

Fpr $u_0 \in \dot{H}^1(\mathbb{R}^3)$, $u_1 \in L^2(\mathbb{R}^3)$ and $f \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^3))$, we admit that this remains true and that the solution has the regularity $\mathcal{C}(\mathbb{R}; \dot{H}^1(\mathbb{R}^3)) \cap \dot{\mathcal{C}}^1(\mathbb{R}; L^2(\mathbb{R}^3))$.

($v$) In this question, we solve the Cauchy problem for $(NLW^\varepsilon)$ with $u_0 \in \dot{H}^1(\mathbb{R}^3)$ and $u_1 \in L^2(\mathbb{R}^3)$. Let $u_L$ be the solution to $(W)$ with $f \equiv 0$ and data $(u_0, u_1)$. Let $\Phi_\varepsilon : v \mapsto w$ with $w$ solution to

$$\begin{cases} \partial_{tt}^2 w - \Delta w = -\varepsilon v^3 \\ (w, \partial_t w)|_{t=0} = (0,0). \end{cases}$$

We note $X_T$ the Banach space $\mathcal{C}([0,T]; \dot{H}^1(\mathbb{R}^3)) \cap \dot{\mathcal{C}}^1([0,T]; L^2(\mathbb{R}^3))$ and $\bar{B}_T(R)$ the closed ball of $X_T$ with center 0 and radius $R$.

(a) Show that there exists $C > 0$ such that

$$\forall v \in \bar{B}_T(R),\ \|\Phi_\varepsilon(v)\|_{X_T} \leq \|(\nabla u_0, u_1)\|_{L^2} + CTR^3$$

and

$$\forall (v,w) \in \bar{B}_T(R) \times \bar{B}_T(R),\ \|\Phi_\varepsilon(v) - \Phi_\varepsilon(w)\|_{X_T} \leq CTR^2 \|v - w\|_{X_T}.$$

(b) Show that there exists $c > 0$ (independant of $u_0, u_1$) such that the map $v \mapsto u_L + \Phi_\varepsilon(v)$ has a unique fixed point in $\bar{B}_{T_0}(R)$ with

$$R = 2\|(\nabla u_0, u_1)\|_{L^2} \quad \text{et} \quad T_0 = \frac{c}{\|(\nabla u_0, u_1)\|_{L^2}^2}.$$

(c) Conclude that $(NLW^\varepsilon)$ has a unique solution in $X_{T_0}$.

(d) Let $T^*$ be the maximal life time of this solution. Show that $T^* < \infty$ implies $\limsup_{t \to T^*} \|\nabla_{t,x} u(t)\|_{L^2} = +\infty$, and then that there exists $C_0 > 0$ such that

$$\|\nabla_{t,x} u(t)\|_{L^2} \geq \frac{C_0}{\sqrt{T^* - t}} \quad \text{for all} \quad t \in [0, T^*[.$$

(e) If $u$ is a smooth solution of $(NLW^\varepsilon)$ in the interval $[0,T]$, show that

$$(E) \qquad \forall t \in [0,T],\ \frac{d}{dt}\Big( \|\partial_t u\|_{L^2}^2 + \|\nabla_x u\|_{L^2}^2 + \frac{\varepsilon}{2}\|u\|_{L^4}^4 \Big)(t) = 0.$$

(f) We admit that $u_0 \in L^4(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3)$ and $u_1 \in L^2(\mathbb{R}^3)$ imply that the maximal solution constructed above satisfies $(E)$. What can we conclude on the life time of the solution in the defocusing case ?

**Exercice 6.5** (Lower bound on the blow up speed)**.** Let $u_0 \in H^1$ and $u \in \mathcal{C}([0,T[, H^1)$ the solution to (6.1) given by Theorem 6.1.1. Let $0 < s_c < 1$ be the scaling parameter given by (6.15). We assume $T < +\infty$. Show that there exists $C(u_0)$ such that for all $t$ close enough to $T$, there holds

$$\|\nabla u(t)\|_{L^2} \geq \frac{C(u_0)}{(T-t)^{\frac{1-s_c}{2}}}.$$

*Hint*: Pick $t_0 \in [0,T[$ and define $v(\tau, x) = (\lambda(t_0))^{\frac{2}{p-1}} u(t_0 + (\lambda(t_0))^2 \tau, \lambda(t_0) x)$ for a well chosen $\lambda(t_0)$.

**Exercice 6.6** (Upper bound on the blow up speed)**.** Let the focusing (NLS)

$$\left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0,x) = u_0(x) \end{array} \right. , \quad x \in \mathbb{R}^2, \quad 3 < p < 5.$$

Let $H^1_r$ be the set of $H^1$ functions with radial symmetry, then the Cauchy problem is well posed in $H^1_r$. We pick $u_0 \in H^1_r$ and assume that the solution blows up in finite time $0 < T < +\infty$. The aim of this problem is to derive an upper bound on $\|\nabla u(t)\|_{L^2}$ as $t \uparrow T$.
Integration by parts should be done without boundary terms (without justification). We let

$$s_c = 1 - \frac{2}{p-1}$$

and $E_0$ be the energy of the data. We recall Hölder:

$$|xy| \leq \frac{1}{p}\left(\frac{|x|}{A}\right)^p + \frac{(A|y|)^{p'}}{p'}, \quad 1 \leq p, p' \leq +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad A > 0. \tag{6.29}$$

(*i*) Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ with spherical symmetry, prove the formulas:

$$\frac{1}{2}\frac{d}{d\tau}\int \chi|u|^2 = Im\left(\int \nabla\chi \cdot \nabla u \overline{u}\right),$$

and

$$\frac{1}{2}\frac{d}{d\tau} Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right) = \int \chi''|\nabla u|^2 - \frac{1}{4}\int \Delta^2\chi|u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\chi|u|^{p+1}.$$

(*ii*) Prove that for all $u \in H^1_r$,

$$\forall R > 0, \quad \|u\|^2_{L^\infty(r\geq R)} \leq \frac{2}{R}\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

(*iii*) Let $R > 0$, $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ with spherical symmetry and

$$\psi(x) = \left| \begin{array}{l} \frac{|x|^2}{2} \text{ pour } |x| \leq 2 \\ 0 \text{ pour } |x| \geq 3 \end{array} \right. .$$

Let

$$\chi(x) = \psi_R(x) = R^2\psi\left(\frac{x}{R}\right),$$

show that

$$c(d,p)\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(d,p)\left[|E_0| + \int_{|x|\geq R}|u|^{p+1} + \frac{1}{R^2}\int_{2R\leq|x|\leq 3R}|u|^2\right]$$

for some constants $c(d,p), C(d,p) > 0$ independant of R.

(*iv*) Prove using (6.29) that:

$$\frac{c(d,p)}{2}\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(u_0,d,p)\left[1 + \frac{1}{R^2} + \frac{1}{R^{\frac{2}{\alpha}}}\right] \tag{6.30}$$

with

$$\alpha = \frac{5-p}{p-1}.$$

(*v*) Integrate in time (6.30) and prove: $\forall 0 < t_0 < t_2 < T$,

$$\int_{t_0}^{t_2}(t_2 - t)\|\nabla u(t)\|^2_{L^2}dt \leq C(u_0,d,p)\left[\frac{(t_2-t_0)^2}{R^{\frac{2}{\alpha}}} + R(t_2-t_0)\|\nabla u(t_0)\|_{L^2} + R^2\right].$$

$(vi)$ Choose $R = (T - t_0)^{\frac{\alpha}{1+\alpha}}$ and conclude that for t close enough to $T$:

$$\int_{t_0}^{T} (T-t)\|\nabla u(t)\|_{L^2}^2 dt \leq C(d,p,u_0)(T-t)^{\frac{\alpha}{2+\alpha}} + (T-t_0)^2 \|\nabla u(t_0)\|_{L^2}^2.$$

$(vii)$ Show that for t close enough to $T$:

$$\int_{t_0}^{T} (T-t)\|\nabla u(t)\|_{L^2}^2 dt \leq C(d,p,u_0)(T-t)^{\frac{2\alpha}{1+\alpha}}.$$

$(viii)$ Conclude that there exists a sequence $t_n \to T$ such that

$$\|\nabla u(t_n)\|_{L^2} \leq \frac{C(d,p,u_0)}{(T-t_n)^{\frac{1}{1+\alpha}}}.$$

(remark: this bound is sharp!).

$(ix)$ Open problem: prove any bound on blow up rate in the critical case $p = 3$!

# Chapter 7

# Variational methods

Let us consider the non linear Schrödinger equation

$$(NLS) \quad \left| \begin{array}{l} i\partial_t u + \Delta u + \varepsilon u|u|^{p-1} = 0 \\ u(t,x) \in \mathbb{C} \\ (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^d). \end{array} \right. \tag{7.1}$$

in dimension $d \geq 1$ and in the focusing case $\varepsilon = 1$. Given $u_0 \in H^1(\mathbb{R}^d)$, we have proved the existence of a unique maximal solution $u \in \mathcal{C}([0,T[;H^1)$ with $T = +\infty$ if $p < 1 + \frac{4}{d}\cdot$ We address the question: what does the solution look like as $t \to +\infty$?

We know the answer in the linear case: solutions disperse to zero at speed which depends on the structure in Fourier of the initial data. For the defocusing case $\varepsilon = -1$ in the subcritical regimes $p < 2^* - 1$, the non linear dynamics is asymptotically attracted by the linear dynamic, and the non linear effect is described by the scattering map. The situation is completely different in the focusing case due to the existence of new solutions: solitons or solitary waves. For (NLS), they take the form of time periodic wave packets $u(t,x) = Q(x)e^{it}$ which therefore do not decay in time. A general conjecture is that all solutions to (7.1) can be decomposed asymptotically in time as trains of decoupled solitary waves coupled to a scattering radiation.

We aim in this chapter at developping various methods for the construction of solitary waves. A classical problem which will guide us is the construction of time periodic solutions to (7.1)

$$u(t,x) = Q(x)e^{it}$$

where the profile $Q$ satisfies:

$$\left| \begin{array}{l} \Delta Q - Q + Q|Q|^{p-1} = 0 \\ Q \in H^1(\mathbb{R}^d). \end{array} \right. \tag{7.2}$$

## 7.1 The variational approach

We introduce the variational setting to study (7.2) and the associated infinite dimensional minimization problem.

### 7.1.1 The space $H^1_r$

Let $d \geq 2$ and consider the space $H^1_r$ which is the subset of functions of $H^1(\mathbb{R}^d;\mathbb{C})$ which have radial symmetry ie

$$u(x) = u(Rx), \quad \forall R \in \mathcal{M}_d(\mathbb{R}) \ \text{ with } \ R^tR = Id,$$

or equivalently

$$u(x) \equiv \tilde{u}(r) \ \text{ with } \ r = |x| = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}} \quad \text{and } \tilde{u} : \mathbb{R}^+ \to \mathbb{C}.$$

This set coincides with the closure of the radially elements of $\mathcal{D}(\mathbb{R}^d)$ for the norm

$$\|u\|^2_{H^1} = \int_{\mathbb{R}^d} \left(|\nabla u(x)|^2 + |u(x)|^2\right) dx = c_d \int_0^{+\infty} \left(|\partial_r \tilde{u}(r)|^2 + |\tilde{u}(r)|^2\right) r^{d-1}\, dr$$

where $c_d$ is the area of the unit sphere of $\mathbb{R}^d$. It is therefore a closed subset of $H^1$. In the sequel , we systematically identify $u : \mathbb{R}^d \to \mathbb{C}$ radial with its representant $\tilde{u} : \mathbb{R}_+ \to \mathbb{C}$.

**Lemma** (Regularity and decay in $H^1_r$)**.** *Let $d \geq 2$ and $u \in H^1_r$, then $u$ belongs to the Hölder space $\mathcal{C}^{\frac{1}{2}}(]0+\infty[;\mathbb{C})$ defined in exercice 2.5 and*

$$\|r^{\frac{d-1}{2}} u\|_{L^\infty} \lesssim \sqrt{\|u\|_{L^2}\|\nabla u\|_{L^2}}. \tag{7.3}$$

*Proof of Lemma 7.1.1.* Let $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^d)$ radial. Then

$$\phi^2(r) = -2\int_r^{+\infty} \phi(\tau)\phi'(\tau)\, d\tau$$

and hence by Cauchy-Schwarz,

$$\phi^2(r) \leq \tfrac{2}{r^{d-1}} \int_r^{+\infty} |\phi(\tau)\phi'(\tau)|\tau^{d-1}\, d\tau \lesssim \frac{1}{r^{d-1}}\|\nabla\phi\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)},$$

and (7.3) is proved. Similarily for $0 < r_1 \leq r_2 < +\infty$,

$$|\phi(r_1) - \phi(r_2)| \;=\; 2\left|\int_{r_1}^{r_2} \phi'(\tau) d\tau\right| \lesssim \frac{1}{r_1^{\frac{d-1}{2}}}\|\phi\|_{H^1}(r_2 - r_1)^{\frac{1}{2}}$$

where we used Cauchy-Schwarz in the first step. The lemma follows by density. $\square$

The decay estimate (7.3) implies the compactness of the radial Sobolev embedding[1]:

**Proposition 7.1.1** (Compactness of the embedding of $H^1_r$ into $L^p$, $2 < p < p_c$)**.** *Let $d \geq 2$ and*

$$p_c \stackrel{def}{=} \begin{cases} +\infty & \text{for } \ d = 2, \\ \frac{2d}{d-2} & \text{for } \ d \geq 3, \end{cases}$$

*then for all $2 < p < p_c$, the embedding $H^1_r \hookrightarrow L^p$ is compact.*

[1] Recall that this is false without the symmetry assumption due to the action of the group of translations.

*Proof of Proposition 7.1.1.* Let $u \in H^1_r$ and $2 < p < p_c$, then (7.3) implies

$$\int_{|x|\geq R} |u|^p\,dx \leq \frac{\|r^{\frac{d-1}{2}}u\|_{L^\infty}^{p-2}}{R^{\frac{(p-2)(d-1)}{2}}}\int_{\mathbb{R}^d}|u|^2\,dx \lesssim \frac{1}{R^{\frac{(p-2)(d-1)}{2}}}\|u\|^p_{H^1}. \tag{7.4}$$

Let $(u_n)_{n\in\mathbb{N}}$ bounded in $H^1_r(\mathbb{R}^d)$, then (7.4) implies that the sequence is $L^p$ tight:

$$\forall\varepsilon > 0,\quad \exists R > 0,\quad \forall n \geq 1,\quad \|u_n\|_{L^p(|x|\geq R)} < \varepsilon. \tag{7.5}$$

Since by Theorem 4.3.2, there exists $u \in H^1(\mathbb{R}^d)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ in } H^1, \quad \text{and} \quad u_n \to u \text{ in } L^p(|x| \leq R),\quad \forall R > 0,$$

we conclude using (7.5),

$$u_n \to u \text{ dans } L^p(\mathbb{R}^d).$$

Since $H^1_r$ is closed and hence weakly closed, $u \in H^1_r$. □

*Remark.* The injection $H^1_r \hookrightarrow L^2$ is never compact due to the action of the dilation group:

$$u_n(r) = \lambda_n^{\frac{d}{2}} u(\lambda_n r),\quad \lambda_n \to 0$$

for some fixed profile $u \in \mathcal{C}^\infty_c$ non nul. One easily checks that $u_n \rightharpoonup 0$ in $H^1$ but $\|u_n\|_{L^2} = \|u\|_{L^2} \neq 0$, and hence no subsequence converges in $L^2$.

### 7.1.2 A compact minimization problem in $H^1_r$

The compactness of the Sobolev embedding allows us to solve infinite dimensional minimization problems.

**Proposition 7.1.2** (Compact minimization)**.** *Let $d \geq 2$ and $p > 1$ satisfying*

$$1 < p < 2^* - 1 = \left|\begin{array}{ll} +\infty & \textit{for}\ \ d = 1,2 \\ \frac{d+2}{d-2} & \textit{for}\ \ d \geq 3. \end{array}\right.$$

*For all $M > 0$, let*

$$\mathcal{A}_M = \left\{ u \in H^1_r \ \ t.q.\ \ \int_{\mathbb{R}^d} |u|^{p+1}\,dx = M \right\}.$$

*Then the minimization problem*

$$I_M = \inf_{u\in\mathcal{A}_M} \|u\|^2_{H^1} \tag{7.6}$$

*has a solution $u_M \in \mathcal{A}_M$.*

*Proof of Proposition 7.1.2.* Since we minimize a positive quantity, we may consider a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathcal{A}_M$ such that

$$\|u_n\|^2_{H^1} \to I_M \geq 0.$$

The sequence $(u_n)_{n\in\mathbb{N}}$ is radial and bounded in $H^1_r$, and hence by Proposition 7.1.1, $(u_n)_{n\in\mathbb{N}}$ converges strongly in $L^{p+1}$ up to a subsequence. Hence there exists $u \in H^1_r$ such that :

$$u_n \to u \text{ in } L^{p+1} \text{ and } u_n \rightharpoonup u \text{ in } H^1.$$

By lower semi continuity of the norm for the weak limit:

$$\|u\|^2_{H^1} \leq \liminf_{n\to+\infty} \|u_n\|^2_{H^1} = I_M$$

and by strong $L^{p+1}$ limit:

$$\|u\|^{p+1}_{L^{p+1}} = \lim_{n\to+\infty} \|u_n\|^{p+1}_{L^{p+1}} = M.$$

Hence $u \in \mathcal{A}_M$ and $\|u\|^2_{H^1} \leq I_M$, hence $u$ attains the infimum. □

## 7.2 Study of the minimizers

We now aim at classifying all the minimizers given by Proposition 7.1.2 and show in particular that they provide radially symmetric solitonic profiles for (NLS).

### 7.2.1 Positivity of the minimizer

**Lemma 7.2.1** (Positivity). *If $u \in \mathcal{A}_M$ minimizes* (7.6)*, then so does* $|u|$*.*

This follows from the following convexity property of the Dirichlet functional

**Lemma 7.2.2** (Convexity estimate for the gradient). *Let $u \in H^1(\mathbb{R}^d;\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^d;\mathbb{R}^+)$ and*

$$\int |\nabla u|^2\, dx \geq \int |\nabla |u||^2\, dx. \tag{7.7}$$

*Moreover, if $u$ is continuous and $\{u \neq 0\}$ is open and connex*[2]*, then the equality holds iff there exists $\gamma \in \mathbb{R}$ such that $u = |u|e^{i\gamma}$.*

*Proof of Lemma 7.2.2.* Decompose $u$ in real and imaginary parts: $u = f + ig$. Then p.p[3]

$$\nabla |u| = \nabla \sqrt{f^2+g^2} = \frac{f\nabla f + g\nabla g}{\sqrt{f^2+g^2}}.$$

Hence

$$\begin{aligned}
\int |\nabla |u||^2\, dx &= \int \frac{|f\nabla f + g\nabla g|^2}{f^2+g^2}\, dx \\
&= \int \frac{1}{f^2+g^2}\left[f^2|\nabla f|^2 + g^2|\nabla g|^2 + 2fg\nabla f\cdot\nabla g\right]\, dx \\
&= \int |\nabla f|^2\, dx + \int |\nabla g|^2\, dx - \int \frac{|g\nabla f - f\nabla g|^2}{f^2+g^2}\, dx,
\end{aligned}$$

which yields (7.7). Assume now $|u| \in H^1(\mathbb{R}^d;\mathbb{R}^+_*)$ with equality in (7.7). Then

$$\text{p.p. } x \in \mathbb{R}^d, \quad f\nabla g = g\nabla f. \tag{7.8}$$

Since $f$ and $g$ are continuous, our assumption ensures that $A \cup B$ with $A \stackrel{\text{def}}{=} f^{-1}(\mathbb{R}\setminus\{0\})$ and $B \stackrel{\text{def}}{=} g^{-1}(\mathbb{R}\setminus\{0\})$ is open and connex. Let $\phi \in \mathcal{D}(B)$. Then

$$h = \frac{\phi}{g} \in H^1(\mathbb{R}^d) \text{ and } \nabla h = \frac{\nabla\phi}{g} - \frac{\nabla g}{g^2}\phi,$$

and we may compute:

$$\begin{aligned}
\int \frac{f}{g}\nabla\phi\, dx &= \int f\nabla\left(\frac{\phi}{g}\right) dx + \int f\phi\frac{\nabla g}{g^2}\, dx = \int f\nabla h\, dx + \int h\frac{f\nabla g}{g}\, dx \\
&= \int f\nabla h\, dx + \int h\nabla f\, dx = 0
\end{aligned}$$

[2] which holds in particular if $u$ does not vanish

[3] This formula can be justified by a regularization argument, see [26], p. 152.

from (7.8). Since this all $\forall \phi \in \mathcal{D}(B)$, we conclude

$$\nabla\left(\frac{f}{g}\right) = 0 \quad \text{in} \quad \mathcal{D}'(B)$$

and hence / $f/g$ is a constant in $B$. The same argument ensures $g/f$ constant in $A$. Since $A \cup B$ is connex, we conclude that $f$ and $g$ are proportional and hence $\exists \gamma \in \mathbb{C}$ such that $u = |u|e^{i\gamma}$. □

### 7.2.2 Euler-Lagrange equations

We are thus left via Lemma 7.2.1 with the classification of the non negative minimizers $u \in H^1_r(\mathbb{R}^d; \mathbb{R}^+)$. The next step is the derivation of the Euler Lagrance equations which tranform into a PDE the minimizing property.

**Proposition 7.2.1** (Euler Lagrange equation)**.** *Let $u \geq 0$ minimizing* (7.6)*, then $\exists \lambda \in \mathbb{R}$ such that*

$$\Delta u - u = -\lambda u^p \quad \textit{dans} \quad H^{-1}. \tag{7.9}$$

*Moreover:*

$$\lambda = \frac{I_M}{M} > 0. \tag{7.10}$$

*Proof of Proposition 7.2.1.* Let $t \in \mathbb{R}$ and $h \in \mathcal{C}^\infty_c(\mathbb{R}^d; \mathbb{R})$ radial, let $u_t = u + th$. We renormalize $u_t$:

$$v_t \stackrel{\text{def}}{=} \frac{\|u\|_{L^{p+1}}}{\|u_t\|_{L^{p+1}}} u_t$$

so that $v_t \in \mathcal{A}_M$. This renormalization makes sense for $t$ small enough since $u$ is not zero. Let us show that $\Phi : t \mapsto \|v_t\|^2_{H^1}$ is derivable at $t = 0$. Since the infinum is attained at $0$, (7.9) will follow by writing $\Phi'(0) = 0$. Let us first compute

$$\|u + th\|^2_{L^2} = \|u\|^2_{L^2} + 2t \int uh\, dx + t^2 \|h\|^2_{L^2}$$

$$\text{and} \qquad \|\nabla(u + th)\|^2_{L^2} = \|\nabla u\|^2_{L^2} + 2t \int \nabla u \cdot \nabla h\, dx + t^2 \|\nabla h\|^2_{L^2}.$$

Then

$$\|u + th\|^2_{L^2} = \|u\|^2_{L^2} + 2t \int uh\, dx + \mathcal{O}(t^2), \tag{7.11}$$

and by integration by parts,

$$\|\nabla(u + th)\|^2_{L^2} = \|\nabla u\|^2_{L^2} - 2t \langle \Delta u, h \rangle_{H^{-1} \times H^1} + \mathcal{O}(t^2). \tag{7.12}$$

Next, to compute the order one term in the development of $\|u_t\|^{p+1}_{L^{p+1}}$, we start with the following homogeneous estimate: forall $q \geq 2$ :

$$||1 + z|^q - 1 - qz| \leq C_q(|z|^2 + |z|^q), \quad \forall z \in \mathbb{R}.$$

Chosing $z = th/u$, integrating and multiplying both terms of the identity by $u^q$ yields

$$\left| \int |u_t|^q\, dx - \int u^q\, dx - q \int th\, u^{q-1}\, dx \right| \leq C_q \left( t^2 \int h^2 u^{q-2}\, dx + t^q \int |h|^q\, dx \right).$$

Let $q = p + 1$, we conclude by Hölder since $h$ is compactly supported:

$$\int |u_t|^{p+1}\,dx = \int |u|^{p+1}\,dx + (p+1)t \int hu^p\,dx + \mathcal{O}(t^2) \quad \text{as} \quad t \to 0. \tag{7.13}$$

Recall the definition of $v_t$, we easily obtain combining (7.11), (7.12) and (7.13) :

$$\|v_t\|_{H^1}^2 = \left(\|u\|_{H^1}^2 + 2t \int uh\,dx - 2t\langle \Delta u, h\rangle_{H^{-1}\times H^1}\right)\left(1 - \frac{2t}{M}\int u^p\,h\,dx\right) + \mathcal{O}(t^2).$$

Hence. for $t$ close enough to $0$,

$$\Phi(t) = \Phi(0) + 2t\left(\int (u - \lambda u^p)h\,dx - \langle \Delta u, h\rangle_{H^{-1}\times H^1}\right) + \mathcal{O}(t^2) \quad \text{with} \quad \lambda \stackrel{\text{def}}{=} \frac{I_M}{M}.$$

We conclude that $\Phi$ is derivable at $0$ with derivative

$$\Phi'(0) = -\langle \Delta u - u + \lambda u^p, h\rangle_{H^{-1}\times H^1}$$

and then

$$\langle \Delta u - u + \lambda u^p, h\rangle_{H^{-1}\times H^1} = 0 \quad \text{for all } h \in \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ radial.}$$

By density, we may extend the above equality to all $h \in H_r^1$, and conclude that (7.9) is satisfied *in the sense of the dual of* $H_r^1$ only. Nevertheless, the distribution $\Delta u - u + \lambda u^p$ is radial, and then decomposing any $h \in C_c^\infty(\mathbb{R}^d)$ as $h = h_r + g$ with $h_r$ radiale and $g$ of zero average on every sphere centered at the origin ensures

$$\int (\Delta u - u + \lambda u^p)\,h\,dx = \int (\Delta u - u + \lambda u^p)\,h_r\,dx = 0,$$

and (7.9) in $H^{-1} = (H^1)^*$, by density. □

### 7.2.3 Regularity and uniqueness of the minimizers

We may now completely classify the family of minimizers. First observe that if $u$ satisfies (7.9) with $\lambda = \lambda(M) > 0$ from (7.10), then

$$v = \lambda^{\frac{1}{p-1}} u$$

satisfies

$$\Delta v - v + v^p = 0, \quad v \geq 0. \tag{7.14}$$

We are therefore left with classifiying the $H_r^1$ positive solutions of (7.14). The equation (7.14) is to be understood in the sense of distributions or in $H^{-1}$ since $v \in H_r^1$. Let us start with showing that $v$ is in fact a smooth and hence a solution in the classical sense.

**Lemma 7.2.3** (Regularity)**.** *The solution $v$ to (7.14) belongs to $\mathcal{C}^2(\mathbb{R}^d)$, tends to $0$ at infinity, and there exists $a > 0$ such that $v$ is the unique solution on $\mathbb{R}^+$ of the Cauchy problem:*

$$\begin{cases} \dfrac{d^2v}{dr^2} + \dfrac{d-1}{r}\dfrac{dv}{dr} = v - v^p, \\ v(0) = a, \quad \dfrac{dv}{dr}(0) = 0. \end{cases} \tag{7.15}$$

*Proof of Lemma 7.2.3.* The regularity of $v$ relies on a bootstrap argument using the smoothing effect of the Laplace operator. For the sake of simplicity, let us restrict the proof to $p = 3$ and $d = 2$, the general case relies on similar arguments. Let us first show that $v \in H^2(\mathbb{R}^2)$. For this, observe that $v \in H^1(\mathbb{R}^2)$ and the Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^6(\mathbb{R}^2)$ ensures $v^3 \in L^2(\mathbb{R}^2)$. Hence (7.14) yields $(\mathrm{Id} - \Delta)v \in L^2$, which through Plancherel ensures $v \in H^2$. Next, we apply $\Delta$ to (7.14) and observe

$$(\mathrm{Id} - \Delta)\Delta v = \Delta(v^3) = 6v|\nabla v|^2 + 3v^2 \Delta v.$$

Since $H^2(\mathbb{R}^2)$ embeds into $L^\infty(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ embeds into $L^4(\mathbb{R}^2)$, we conclude that the right hand side belongs to $L^2$. We conclude that $\Delta v \in H^2$, and hence $v \in H^4$. Sobolev injections then ensure that $v$ and all its derivatives of order 1 and 2 are continuous and tend to 0 at infinity. In particular, $v \in \mathcal{C}^2(\mathbb{R}^2)$. Since $v$ is radial, computing the $\Delta$ in spherical coordinates ensures

$$\forall r > 0, \quad \frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = v - v^3. \tag{7.16}$$

The $\mathcal{C}^2$ regularity of $v$ allows us to conclude

$$v'(r) \to 0 \quad \text{as} \quad r \to 0$$

and hence $v$ satisfies (7.15) for some $a \geq 0$. Finally, the fact that the Cauchy problem (7.15) is locally well posed follows by remarking

$$\frac{d^2 v}{dr^2} + \frac{d-1}{r}\frac{dv}{dr} = \frac{1}{r^{d-1}}\frac{d}{dr}\left(r^{d-1}\frac{dv}{dr}\right)$$

and hence we solve by fixed point in$\mathcal{C}^1([0, R])$, $R = R(a) > 0$ small enough, the corresponding integral equation

$$v(r) = a + \int_0^r \left(\frac{ds}{s^{d-1}} \int_0^s \tau^{d-1} f(v(\tau)) d\tau\right) \quad \text{with} \quad f(v) = v - v^p.$$

Since for $a = 0$, $u = 0$ is a solution and hence the unique solution, we conclude that $a > 0$ and the Lemma is proved. □

We have therefore reduced the understanding of positive $H^1_r$ solutions of the PDE (7.14) to the description of solutions to the one dimensional ODE , indexed by the *shooting parameter* $a$. This is not a trivial problem and after the first proof of Kwong [24] in 1987, a simplified canonical proof is proposed in MacLeod [28], see also the appendix of [39].

**Theoreme 7.2.1** (Uniqueness in the sense of the dynamical systems)**.** *There exists a unique* $a > 0$ *such that the corresponding solution* $v(r)$ *to* (7.15) *satisfies*

$$\forall r > 0, \quad v(r) \geq 0$$

*and the boundary condition*

$$v(r) \to 0 \quad \textit{when} \quad r \to +\infty. \tag{7.17}$$

*Moreover,*

$$\forall r \geq 0, \quad v(r) > 0. \tag{7.18}$$

*We note* $Q(r)$ *this solution: it is the* ground state *of* (7.15).

**Remark 7.2.1.** *Let us stress that the strict positivity* (7.18) *follows from Cauchy-Lipschits: if* $Q(r_0) = 0$, *then* $Q'(r_0) = 0$ *since* $Q$ *is positive, and hence* $Q \equiv 0$. *Note also that Theorem 7.2.1 is trivial in dimension* $d = 1$ *where all solutions can be computed explicitely (see exercice 7.1).*

### 7.2.4 Conclusion

Let us summarize. Let $u \in H^1_r$ a minimizer of (7.6), then $|u|$ is a positive minimizer by Lemma 7.2.1. Then by Proposition 7.2.1,

$$v \stackrel{\text{def}}{=} \lambda^{\frac{1}{p-1}} |u| \text{ with } \lambda = \frac{I_M}{M} > 0$$

solves

$$\Delta v - v + v^p = 0, \quad v \geq 0, \quad v \in H^1_r.$$

Hence by Lemma 7.2.3, $v$ is a strong positive solution of (7.15). Moreover, $v \in H^1_r$ implies

$$v(r) \to 0 \text{ when } r \to +\infty$$

by (7.3), and hence Theorem 7.2.1 implies

$$v(r) = Q(r).$$

Hence $|u|$ is continuous and does not vanish. Finally, $u$ and $|u|$ being both minimizers, we are in the equality case of Lemma 7.2.2, and hence

$$u = |u|e^{i\gamma}, \quad \gamma \in \mathbb{R}.$$

We have proved:

**Proposition 7.2.2** (Classification of minimizers). *Let $M > 0$ and*

$$\mathcal{A}_M = \left\{ u \in H^1_r \;\; \textit{with} \;\; \int_{\mathbb{R}^d} |u|^{p+1}\, dx = M \right\},$$

*then the minimization problem*

$$I_M = \inf_{u \in \mathcal{A}_M} \|u\|^2_{H^1}$$

*is attained exactly on the one parameter family*

$$e^{i\gamma} \left( \frac{M}{I_M} \right)^{\frac{1}{p-1}} Q(r), \quad \gamma \in \mathbb{R},$$

*where $Q$ is the unique ground state solution given by Theorem 7.2.1.*

## 7.3 Exercices

**Exercice 7.1** (Computing the ground state in dimension $d = 1$). Let $1 < p < +\infty$.

(*i*) Let $a \in \mathbb{R}^+$. Show that there exists a unique maximal solution $u \in \mathcal{C}^1(]R_1, R_2[; \mathbb{R})$ of the non linear ODE:
$$\begin{cases} Q'' - Q + Q^p = 0, \\ Q(0) = a, \quad Q'(0) = 0. \end{cases}$$

(*ii*) Compute a first integral (hint: multiply by $Q'$).

(*iii*) Show that there exists at most one global solution $Q$ which goes to 0 as $+\infty$, and that this solution satisfies
$$\forall x > 0, \quad Q(x) > 0 \text{ and } Q'(x) < 0.$$

$(iv)$ Change variables $y = \frac{1}{Q^{\frac{p-1}{2}}}$ and obtain the ground state formula

$$Q(x) = \left( \frac{p+1}{2\cosh^2\left( (\frac{p-1}{2})x \right)} \right)^{\frac{1}{p-1}}.$$

**Exercice 7.2.** Let $V : \mathbb{R}^d \to \mathbb{R}$ continuous with $\lim_{|x|\to+\infty} V(x) = 0$.

$(i)$ Show that for all $s > 0$, the operator $T : u \to Vu$ is compact from $H^s(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$.

$(ii)$ Let $E_{min}$ be the energy of the ground state of the Schrödinger operator $-\Delta - V$:

$$E_{min} \stackrel{\text{def}}{=} \inf\{E(u),\ \ u \in H^1(\mathbb{R}^d),\ \ \|u\|_{L^2(\mathbb{R}^d)} = 1\} \ \text{ où } \ E(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_{\mathbb{R}^d} V|u|^2 \, dx.$$

Assume $E_{min} < 0$. Show that $E_{min}$ is attained :

$$\exists u \in H^1(\mathbb{R}^d) \ \text{ tel que } \ \|u\|_{L^2(\mathbb{R}^d)} = 1 \ \text{ et } \ E(u) = E_{min},$$

and that every minimizing sequence converges *strongly* in $H^1(\mathbb{R}^d)$.

$(iii)$ Conclude that there exists an eigenmode for $-\Delta - V$: $\exists \lambda < 0$ and $u \in H^1(\mathbb{R}^d)$ non zero such that

$$-\Delta u - Vu = \lambda u, \quad u(x) \geq 0. \tag{7.19}$$

$(iv)$ Show that $u$ belongs to all Sobolev spaces $H^s$, and that it is a classical solution to (7.19).

$(v)$ We now assume that $V \geq 0$ for the rest of the exercice. Let $p \in ]1, 1 + 4/d[$ and $F$ be the non linear functional defined by

$$F(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^d} V(x)|u(x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |u(x)|^{p+1} \, dx.$$

Let the minimization problem for $M > 0$:

$$I(M) = \inf_{u \in H^1_r(\mathbb{R}^d)} \Big\{ F(u), \ \ \int_{\mathbb{R}^d} |u(x)|^2 \, dx = M \Big\}.$$

Show that

$$I < 0.$$

$(vi)$ Show that all minimizing sequences are bounded in $H^1(\mathbb{R}^d)$.

$(vii)$ We assume that $I(M)$ is a strictly dereasing function of M. Show that $I(M)$ is attained.

$(viii)$ Assume moreove $V$ radial, conclude that there exists a non linear eigenmode: $\exists \lambda \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^d)$ such that

$$-\Delta u - Vu - u^p = \lambda u, \quad u(x) \geq 0.$$

**Exercice 7.3** (One dimensional Hardy inequality)**.** We work on $\mathbb{R}$ and all functions are considered real valued.

$(i)$ Let

$$A = \left\{ u \in \mathcal{C}_c^\infty(\mathbb{R};\mathbb{R}), \quad \int_{\mathbb{R}} \frac{|u(x)|^2}{1+x^2}\, dx = 1 \right\}.$$

Show that

$$\inf_{u \in A} \int_{\mathbb{R}} |u'(x)|^2\, dx = 0.$$

Hint: consider

$$u_n(x) \stackrel{\text{def}}{=} \chi\left(\frac{x}{n}\right) \quad \text{with} \quad \chi(x) = \begin{cases} 1 & \text{for} \quad |x| \leq 1 \\ 0 & \text{pour} \quad |x| \geq 2. \end{cases}$$

$(ii)$ Show that there exists a universal constant $c_1 > 0$ such that

$$\forall u \in \mathcal{C}_c^\infty(\mathbb{R}), \quad u^2(1) + \int_{x \geq 1} |u'(x)|^2\, dx \geq c_1 \int_{x \geq 1} \frac{|u(x)|^2}{x^2}\, dx.$$

Hint: integrate by parts in

$$\int_{x \geq 1} \frac{u^2}{x^2}\, dx = -\int_{x \geq 1} u^2 \left(\frac{1}{x}\right)'\, dx$$

and use

$$2|xy| \leq \frac{x^2}{A} + Ay^2, \quad \forall A > 0.$$

$(iii)$ Show that there exists a universal constant $c_2 > 0$ such that

$$u^2(1) + \int_{|x| \leq 1} |u'(x)|^2\, dx \geq c_2 \left[ u^2(-1) + \int_{|x| \leq 1} \frac{|u(x)|^2}{1+x^2}\, dx \right].$$

$(iv)$ Conclude that there exists a universal constant $c_3 > 0$ such that

$$\forall u \in \mathcal{C}_c^\infty(\mathbb{R}), \quad u^2(1) + \int_{\mathbb{R}} |u'(x)|^2\, dx \geq c_3 \int_{\mathbb{R}} \frac{u^2(x)}{1+x^2}\, dx.$$

$(v)$ Fix $\psi \in \mathcal{D}(\mathbb{R})$ with

$$\int_{\mathbb{R}} \psi(x)\, dx \neq 0 \text{ and } \quad \psi(x) = 0 \quad \text{for} \quad |x| \geq 1.$$

Let

$$A_\psi = \left\{ u \in \mathcal{C}_c^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} \frac{u^2(x)}{1+x^2}\, dx = 1 \quad \text{and} \quad \int_{\mathbb{R}} u(x)\psi(x)\, dx = 0 \right\}.$$

We want to prove:

$$I_\psi = \inf_{u \in A_\psi} \int_{\mathbb{R}} |u'(x)|^2\, dx > 0. \tag{7.20}$$

We argue by contradiction and assume $I_\psi = 0$. Let $u_n \in A_\psi$ with

$$\int_{\mathbb{R}} |u_n'(x)|^2\, dx \leq \frac{1}{n}.$$

Show that

$$\liminf_{n \to +\infty} u_n^2(1) > 0.$$

$(vi)$ Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ with

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| \ge 2 \end{cases} \quad \text{and} \quad \chi(x) > 0 \quad \text{for} \quad 1 \le |x| \le 2.$$

Let

$$v_n(x) = \chi(x) u_n(x).$$

Show that $(v_n)_{n \ge 1}$ is bounded in $H^1(\mathbb{R})$ and there exists a subsequence

$$\begin{cases} v_{\phi(n)} \rightharpoonup v & \text{dans} \quad H^1(\mathbb{R}) \\ v_{\phi(n)} \to v & \text{in} \quad L^\infty(|x| \le 2). \end{cases}$$

$(vii)$ Show that $\int_{\mathbb{R}} v\psi \, dx = 0$.

$(viii)$ Show that $v(1) \neq 0$.

$(ix)$ Let $A > 0$ and $\chi_A(x) = \chi\left(\frac{x}{A}\right)$. Show that

$$\int \chi_A |v' - v_n'|^2 \, dx = \int \chi_A |v_n'|^2 \, dx - \int \chi_A |v'|^2 \, dx - 2 \int \chi_A v'(v_n' - v') \, dx.$$

Conclude that

$$\forall 0 < A < \frac{1}{2}, \quad \int \chi_A |v'|^2 \, dx = 0.$$

$(x)$ Conclude the proof of (7.20).

# Chapter 8

# More solitary waves

The varational approach developped in chapter 7 is a powerful tool but does not always apply: the non linearity has no reason in general to be a gradient. We present in this chapter two other powerful methods which have applications everywhere in mathematical physics: the Lyapounov-Schmidt bifurcation argument, and the direct ODE approach.

## 8.1 The Lyapounov Schmidt bifurcation method

The Lyapounov Schmidt bifurcation argument is *the* canonical method to start *bifurcation* branches and is at the heart of perturbation theory. We shall illustrate the method on an elementary problem related to the harmonic oscillator, but the method goes very far beyond and is probably the most powerful known tool to construct nonlinear objects and solitons.

### 8.1.1 The resolvent of the harmonic oscillator

Let us consider the harmonic oscillator operator

$$H = -\Delta u + (1 + |x|^2)u$$

where we may without loss of generality assume that all functions are real valued. Let us consider the virial space

$$\Sigma = \{u \in H^1(\mathbb{R}^d), \quad xu \in L^2(\mathbb{R}^d)\}$$

which is a Hilbert space for the natural scalar product

$$\langle u, v\rangle_\Sigma = \langle u, v\rangle_{H^1} + \langle |x|^2 u, v\rangle_{L^2}.$$

The solutions to the eigenvalue problem

$$Hu = \lambda u, \quad u \in \Sigma$$

correspond to the energy levels acquired by a quantum particle trapped by the $|x|^2$ magnetic field. The functional setting to study $H$ is the following which adresses the *resolvent* of $H$.

Let us start with the following elementary observation which we will systematically use in the sequel.

**Lemma 8.1.1.** *Let $u \in \Sigma$ and $f \in L^2$ such that*

$$Hu = f \quad in \quad \mathcal{D}'.$$

*Then*

$$\forall v \in \Sigma, \quad \int \nabla u \cdot \nabla v + \int (1+|x|^2)uv = \int uf. \tag{8.1}$$

*Proof of Lemma 8.1.1.* Let $v \in \Sigma$, then since $\mathcal{D}$ is dense in $\Sigma$, there exists $\phi_n \in \mathcal{D}$ such that $\phi_n \to v$ in $\Sigma$. Hence

$$\begin{aligned}
&\int \nabla u \cdot \nabla v + \int (1+|x|^2)uv - \int uf = \lim_{n\to+\infty} \left[\int \nabla u \cdot \nabla \phi_n + \int (1+|x|^2)u\phi_n - \int uf\right] \\
= \quad & \lim_{n\to+\infty} (Hu - f, \phi_n)_{\mathcal{D}',\mathcal{D}} = 0.
\end{aligned}$$

□

We start with the study of the *resolvent* of $H$.

**Proposition 8.1.1** (Resolvent of the harmonic oscillator)**.** *For all $f \in L^2(\mathbb{R}^d)$, there exists a unique $u \in \Sigma$ such that*

$$Hu = f \quad in \quad \mathcal{D}'(\mathbb{R}^d). \tag{8.2}$$

*Moreover, the resolvent map $T(f) = u$ is continuous from $L^2$ into $\Sigma$, injective, and compact and self adjoint from $L^2$ into itself.*

*Proof of Proposition 8.1.1.* The proof relies on the Lax Milgram approach.

**step 1** Existence and continuity of the resolvent. Let $f \in L^2(\mathbb{R}^d)$. Consider the linear form:

$$L_f(v) = \langle f, v\rangle_{L^2},$$

then from Cauchy Schwarz

$$\forall v \in \Sigma, \quad |L_f(v)| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{\Sigma},$$

and hence $L_f$ is continuous linear form on the Hilbert space $\Sigma$. Riesz representation Theorem ensures that there exists a unique $u = T(f) \in \Sigma$ such that

$$\forall v \in \Sigma, \quad \langle u, v\rangle_{\Sigma} = L_f(v) = \langle f, v\rangle_{L^2}. \tag{8.3}$$

In particular for $v = \phi \in \mathcal{D}(\mathbb{R}^d)$:

$$\begin{aligned}
&\langle f, \phi\rangle_{L^2} = \langle u, \phi\rangle_{\Sigma} = \int \nabla u \cdot \nabla \phi + \int (1+|x|^2)u\phi \\
\Leftrightarrow \quad & \left(-\Delta u + (1+|x|^2)u - f, \phi\right)_{\mathcal{D}',\mathcal{D}} = 0
\end{aligned}$$

and hence $u \in \Sigma$ solves (8.2). Moreover applying (8.3) with $v = u$ ensures

$$\|u\|_{\Sigma}^2 = \langle f, u\rangle_{L^2} \leq \|f\|_{L^2}\|u\|_{L^2} \leq \|f\|_{L^2}\|u\|_{\Sigma} \Rightarrow \|u\|_{\Sigma} \leq \|f\|_{L^2}.$$

We now claim that there exists a unique $u \in \Sigma$ solution to (8.2). Indeed, by linearity, let $u \in \Sigma$ with $Hu = 0$ in $\mathcal{D}'$, then applying (8.1) with $v = u$ and $u = 0$ yields $\|u\|_{\Sigma}^2 = 0$ and hence $u = 0$. This concludes the proof of the existence and continuity of the resolvent operator

$T$ as a linear map from $L^2$ into $\Sigma$, and its injectivity.

**step 2** Self adjointness. Since $T \in (L^2, \Sigma)$ and $\|u\|_{L^2} \leq \|u\|_\Sigma$, we conclude that $T \in (L^2, L^2)$. We now claim that $T$ is self adjoint as an endomorphism of $L^2$. Indeed, let

$$Tu = f, \quad Tv = g, \quad (u,v)\Sigma \times \Sigma, \quad (f,g) \in L^2 \times L^2,$$

then from (8.3):

$$\left| \begin{array}{l} \langle Tu, v\rangle_{L^2} = \langle f, v\rangle_{L^2} = \langle u, v\rangle_\Sigma \\ \langle u, Tv\rangle_{L^2} = \langle Tv, u\rangle_{L^2} = \langle g, u\rangle_{L^2} = \langle u, v\rangle_\Sigma \end{array} \right.$$

and since we assumed a real Hilbertian structure[1]

$$\langle u, v\rangle_\Sigma = \langle u, v\rangle_\Sigma \Rightarrow \langle Tu, v\rangle_{L^2} = \langle u, Tv\rangle_{L^2} \Rightarrow T^* = T.$$

**step 3** Compactness. We now claim that $T$ is a compact endormorphism of $L^2$. Indeed, let $f_n \rightharpoonup 0$ in $L^2$, then $u_n = Tf_n$ is bounded in $\Sigma$. Moreover $T \in \mathcal{L}(L^2, \Sigma)$ and hence is weakly continuous, Proposition 2.2.3, which ensures

$$u_n \rightharpoonup 0 \text{ in } \Sigma \Rightarrow u_n \rightharpoonup 0 \text{ in } H^1 \tag{8.4}$$

where we applied Remark 2.2.2. Pick $\varepsilon > 0$. Since $u_n$ is bounded in $\Sigma$, there exists $R(\varepsilon)$ such that

$$\forall n \geq 1, \quad \int_{|x| \geq R(\varepsilon)} |u_n|^2 \leq \frac{1}{R^2(\varepsilon)} \int_{|x|\geq R(\varepsilon)} |x|^2 |u_n|^2 \leq \frac{C}{R^2(\varepsilon)} \leq \varepsilon.$$

On the other hand from (8.4) and Rellich's Theorem 4.3.2, we have

$$\lim_{n \to +\infty} \int_{|x| \leq R(\varepsilon)} |u_n|^2 = 0.$$

We have proved

$$f_n \rightharpoonup 0 \text{ in } L^2 \Rightarrow Tf_n \to 0 \text{ in } L^2$$

and hence $T$ is compact as an endomorphism of $L^2$ by Proposition 2.2.4. □

**Remark 8.1.1.** *Note that the above proof shows that the embedding $\Sigma \subset L^2$ is compact.*

### 8.1.2 Diagonalization of the harmonic oscillator

The *spectral Theorem* asserts that a self adjoint *compact* operator is diagonalizable in a Hilbertian basis.

**Theoreme 8.1.1** (Spectral theorem)**.** *Let $T$ be a self adjoint compact endomorphism of a separable Hilbert space $H$, then $T$ is diagonalizable in a Hilbertian basis of $H$.*

In the case of the harmonic oscillator, eigenvalues and eigenvectors can in fact be computed explicitely using the Hermite functions $\psi_n(x)$.

**Proposition 8.1.2** (Diagonalization of $H$)**.** *There exists an increasing sequence $(\lambda_n \in \mathbb{R}_+^*)_{n\geq 0}$ with*

$$\lim_{n\to\infty} \lambda_n = +\infty$$

*such that the Hermite functions $(\psi_n)_{n\geq 1} \in \Sigma$ is a Hilbertian basis of $L^2$ with*

$$\forall n \geq 0, \quad H\psi_n = \lambda_n \psi_n.$$

[1] the same can be proved for complex valued functions as well.

**Remark 8.1.2.** *Note that* $T\psi_n = \frac{\psi_n}{\lambda_n}$ *and hence* $\frac{1}{\lambda_n} \to 0$ *is the sequence of eigenvalues of* $T$ *on* $L^2$.

*Proof.* The key step is the conjugation formula:

$$\left| \begin{array}{l} u = e^{-\frac{|x|^2}{2}} v \Rightarrow Hu = e^{-\frac{|x|^2}{2}} \left(-\Delta + (d+1) + 2\Lambda\right) v \\ \Lambda v = x \cdot \nabla v. \end{array} \right. \tag{8.5}$$

Indeed, we compute

$$\left| \begin{array}{l} u = e^{-\frac{\alpha x^2}{2}} v \\ \nabla u = e^{-\frac{\alpha |x|^2}{2}} \left[-\alpha x v + \nabla v\right] \\ \Delta = e^{-\frac{\alpha |x|^2}{2}} \left[-\alpha d v - \alpha x \cdot \nabla v + \Delta v - \alpha x \cdot (-\alpha x v + \nabla v)\right] = e^{-\frac{\alpha |x|^2}{2}} \left[\Delta v - 2\alpha x \cdot \nabla v - d\alpha v + \alpha^2 |x|^2 v\right] \end{array} \right.$$

and hence

$$\begin{array}{rcl} Hu & = & e^{-\frac{\alpha |x|^2}{2}} \left[\Delta v - 2\alpha x \cdot \nabla v - d\alpha v + \alpha^2 |x|^2 v + (1+x^2) v\right] \\ & = & e^{-\frac{\alpha |x|^2}{2}} \left[-\Delta v + 2\alpha x \cdot \nabla v + (d\alpha + 1) v + (1-\alpha^2) x^2 v\right] \end{array}$$

and hence $\alpha = 1$ yields (8.5). Let us give the proof of Proposition 8.1.2 in dimension $d = 1$. The case $d \geq 2$ follows by splitting any $u \in \Sigma$ in spherical harmonics.
For $d = 1$, the operator $\tilde{H} v$ is trivial to diagonalize using polynomials. Indeed,

$$\tilde{H} 1 = 2 \Leftrightarrow \left| \begin{array}{l} H\psi_0 = \lambda_0 \psi_0 \\ \lambda_0 = 2, \quad \psi_0 = e^{-\frac{x^2}{2}}. \end{array} \right.$$

Then since

$$\Lambda x^n = n x^n,$$

one easily constructs a polynomial $P_n = x^n + \sum_{k=0}^{n-1} a_k x^k$ such that

$$(\tilde{H} - \lambda_n) P_n = 0 \quad \text{for} \quad \lambda_n = 2n + 2,$$

and hence

$$H\psi_n = \lambda_n \psi_n, \quad \left| \begin{array}{l} \lambda_n = 2n + 2 \\ \psi_n = P_n e^{-\frac{x^2}{2}}. \end{array} \right.$$

The polynomial $P_n$ is the Hermite polynomial of degree $n \geq 0$ and $\psi_n$ is the associated Hermite function. Note that the family $\psi_n$ is orthogonal since $\psi_n$ is an eigenvector for $T$ which is self adjoint on $H = L^2$, and moreover since by Stone Weierstrass polynomials are dense in $\mathcal{C}(|x| \leq R)$ for any $R$, the family $(\psi_n)_{n \geq 0}$ is total[2] in $L^2$ and hence it is up to normalization a Hilbertian basis of $L^2$ in which $T$ is diagonal, and hence $H$. □

### 8.1.3 Varational characterization of the first eigenvalue

We now characterize variationally the first eigenvalue and prove $H - \lambda_0$ has continuous resolvent when restricted to the subspace of functions orthogonal to the first eigenvalue. The proof below is canonical and can be applied to a large class of operators.

[2] ie the vectorial space of finite linear combinations of the $\psi_n$ is dense in $L^2$

**Proposition 8.1.3** (Variational characterization of the first eigenvalue)**.** *Let $d \geq 1$.*
1. Variational characterization. *The minimization problem*

$$I = \inf_{u \in A} \|u\|_{\Sigma}^2, \quad A = \{u \in \Sigma, \|u\|_{L^2} = 1\}$$

*is attained exactly on*

$$I = d+1, \quad u \in \text{span}\{\psi_0 = e^{-\frac{|x|^2}{2}}\}.$$

*Moreover, $I = \lambda_0$ where $\frac{1}{\lambda_0}$ is the largest eigenvalue of $T$.*
2. Resolvent of $H - \lambda_0$. *There exists $c > 0$ such that $\forall f \in L^2$ with $\langle f, \psi_0 \rangle_{L^2} = 0$, there exists a unique $u \in \Sigma$ with $\langle u, \psi_0 \rangle_{L^2} = 0$ such that $(H - \lambda_0)u = f$ and $\|u\|_{\Sigma} \leq C\|f\|_{L^2}$.*

**Remark 8.1.3.** *The first statement implies that the first eigenvalue of $T$ is simple which is a special case of the Krein-Rutman Theorem. For the second statement, observe that if $(H - \lambda_0)u = f$ in $\mathcal{D}'$, then from* (8.1) *and $(H - \lambda_0)\psi_0 = 0$*

$$\left| \begin{array}{l} \int \nabla u \cdot \nabla \psi_0 + \int (1+|x|^2) u \psi_0 = \lambda_0 \int u \psi_0 + \int \psi_0 f \\ \int \nabla \psi \cdot \nabla u + \int (1+|x|^2) \psi_0 u = \lambda_0 \int \psi_0 u \end{array} \right. \Rightarrow \int \psi_0 f = 0$$

*and hence $\langle f, \psi_0 \rangle_{L^2} = 0$ is a necessary condition for the equation to be solvable in $\Sigma$. Proposition 8.1.3 says that this condition is necessary and sufficient which is a special case of the Fredholm alternative.*

*Proof of Proposition 8.1.3.* This follows from an elementary variational argument.

**step 1** Compactness of the minimization problem. Consider the minimization problem

$$I = \inf_{u \in A} \|u\|_{\Sigma}^2, \quad A = \{u \in \Sigma, \|u\|_{L^2} = 1\}.$$

Then $I \geq 0$ by definition and we may consider a minimizing sequence which is bounded in $\Sigma$. Recalling Remark 8.1.1, we have up to a subsequence

$$\left| \begin{array}{l} u_n \rightharpoonup u \ \text{ in } \ \Sigma \\ u_n \to u \ \text{ in } \ L^2 \end{array} \right.$$

and hence $u \in \Sigma$, $\|u\|_{L^2} = \lim_{n \to +\infty} \|u_n\|_{L^2} = 1$ ensures $u \in A$ and

$$\|u\|_{\Sigma}^2 \leq \liminf_{n \to +\infty} \|u\|_{\Sigma}^2 = I$$

and hence $u$ attains the infimum. Arguing verbatim like for the proof of (7.9), we conclude that there exists $\mu \in \mathbb{R}$ such that

$$Hu = \mu u \in \mathcal{D}'$$

and then from (8.1):

$$\|u\|_{\Sigma}^2 = \mu \int |u|^2 = \mu \Leftrightarrow \mu = I.$$

Note that $I > 0$ since otherwise $u \equiv 0$ contradicts $\|u\|_{L^2} = 1$. Moreover by (7.7), $|u|$ is also a minimizer and thus we may without loss of generality assume $u \geq 0$ and $u \neq 0$.

**step 2** Uniqueness of the minimizer. Let now $\phi(x) = e^{-\frac{|x|^2}{2}}$, then from (8.5):

$$H\phi = (d+1)\phi,$$

and then using (8.1) and $Hu = Iu$, $u \geq 0$, ensures:

$$\left| \begin{array}{l} \int \nabla\phi \cdot \nabla u + \int (1+|x|^2)\phi u = (d+1)\int \phi u \\ \int \nabla u \cdot \nabla \phi + \int (1+|x|^2) u\phi = I \int u\phi \end{array} \right. \Rightarrow [I-(d+1)]\int \phi u = 0 \Rightarrow I = d+1$$

since $\phi > 0, u \geq 0$ and $u \neq 0$. Hence $\psi_0, u$ are both minimizers, and hence using (7.7), so is the function $\sqrt{\frac{\psi_0^2+u^2}{2}}$ with necessarily the same Dirichlet energy, and hence since it does not vanish, we are in the equality case of Lemma 7.2.2 and hence $u, \psi_0$ are proportional.

**step 3** Computation of $\lambda_0$. It remains to show that $I = \frac{1}{\lambda_0}$ where $\frac{1}{\lambda_0}$ is the largest eigenvalue of $T$. Indeed, let $T\phi_0 = \frac{1}{\lambda_0}\phi_0$ with $\|\phi_0\|_{L^2} = 1$, then $H\phi_0 = \lambda_0\phi_0$ and hence $\|\phi_0\|_\Sigma^2 = \lambda_0$ implies $\lambda_0 \geq I$. On the other hand, $H\psi_0 = I\psi_0$ implies $T\psi_0 = \frac{\psi_0}{I}$ and hence $\frac{1}{I} \leq \frac{1}{\lambda_0}$ implies $I \geq \lambda_0$.

**step 4** Coercivity of $H - \lambda_0$. Let us now consider the minimization problem

$$J = \inf_{u \in B} \|u\|_\Sigma^2, \quad B = \{u \in \Sigma, \quad \|u\|_{L^2} = 1, \quad \langle u, \psi_0\rangle_{L^2} = 1\}.$$

A minimizing sequence is bounded in $\Sigma$ and hence up to a subsequence $u_n \rightharpoonup u$ in $\Sigma$ and $u_n \to u$ in $L^2$ which ensures

$$u \in B \quad \text{and} \quad \|u\|_\Sigma^2 \leq J$$

and hence $u$ attains the infimum. The Lagrange multiplier argument ensures that there exists $\mu_1, \mu_2 \in \mathbb{R}$ such that

$$Hu = \mu_1 u + \mu_2 \psi_0 \quad \text{in} \quad \mathcal{D}'.$$

Using $\langle u, \psi_0\rangle_{L^2} = 0$ and (8.1) yields $\mu_2 = 0$ and hence $u$ is a eigenvalue of $H$ with $\mu_1 = J$, and hence and eigenvalue of $T$. Since $\langle u, \psi_0\rangle_{L^2} = 0$ and the first eigenvalue is simple, necessarily $\mu_1 = J \geq \lambda_1 > \lambda_0$. By linearity, we have proved that there exists $c > 0$ such that

$$\forall u \in \Sigma \quad \text{with} \quad \langle u, \psi_0\rangle_{L^2} = 0, \quad \|u\|_\Sigma^2 \geq (c+\lambda_0)\|u\|_{L^2}^2. \tag{8.6}$$

**step 5** Resolvent of $H - \lambda_0$. Let now $f \in L^2$ with $\langle f, \psi_0\rangle_{L^2} = 0$, we consider the minimization problem

$$K = \inf_{u \in B} \|u\|_\Sigma^2 - \lambda_0\|u\|_{L^2}^2 - \langle f, u\rangle_{L^2}, \quad C = \{u \in \Sigma, \quad \langle u, \psi_0\rangle_{L^2} = 0\}.$$

Then from (8.6). for $u \in C$,

$$\|u\|_\Sigma^2 - \langle f, u\rangle_{L^2} \geq \|u\|_\Sigma^2 - \|f\|_{L^2}\|u\|_{L^2} \geq c\|u\|_\Sigma^2 - \frac{1}{c}\|f\|_{L^2}$$

for some $c > 0$, and hence $K \geq -c\|f\|_{L^2}^2 > -\infty$ and every minimizing sequence is bounded in $\Sigma$. Hence up to a subsequence $u_n \rightharpoonup u$ in $\Sigma$ and $u_n \to u$ in $L^2$, and hence $u \in C$ and ensures

$$K = \lim_{n \to +\infty} \|u_n\|_\Sigma^2 - \langle f, u_n\rangle_{L^2} \geq \|u\|_\Sigma^2 - \langle f, u\rangle_{L^2}$$

and hence $u$ attains the infimum. Lagrange multiplier yields $Hu - \lambda_0 u - f = 0$ in $\mathcal{D}'$ with $u \in \Sigma$. Uniqueness follows immediately from (8.6) and Proposition 8.1.3 is proved. $\square$

### 8.1.4 The bifurcation branch

The explicit diagonalization of $H$ yields the energy level of a quantum particle trapped in a magnetic potential. The basic problem in the theory of perturbations is the following: say we add a potential $V(x) \in L^\infty(\mathbb{R}^d)$, and we pick a small $\varepsilon > 0$, how do we compute the first eigenvalue of the deformed operator $H_\varepsilon = H + \varepsilon V$? The general Lyapounov Schmidt argument answers precisely this question, and the framework is extremely general.

**Proposition 8.1.4** (Perturbation theory via Lyapounov Schmidt)**.** *Let $V \in L^\infty(\mathbb{R}^d)$, then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$, there exists $\psi_\varepsilon \in \Sigma$ and $\lambda_\varepsilon$ with $|\lambda - \lambda_0| \lesssim \varepsilon$ with*

$$H_\varepsilon \psi_\varepsilon = \lambda_\varepsilon \psi_e.$$

*Moreover,*

$$\lambda = \lambda_0 + \varepsilon \frac{\langle V\psi_0, \psi_0\rangle_{L^2}}{\|\psi_0\|_{L^2}} + O(\varepsilon^2). \tag{8.7}$$

*Proof of Proposition 8.1.4.* Let us look for a solution to $H_\varepsilon \psi_\varepsilon = \lambda_e \psi_\varepsilon$ in the form

$$\psi_\varepsilon = \psi_0 + \varepsilon \psi_1, \quad \lambda_\varepsilon = \lambda_0 + \varepsilon \lambda_1.$$

Then using $H\psi_0 = \lambda_0 \psi_0$:

$$\begin{aligned}
& H_\varepsilon \psi_\varepsilon = \lambda_e \psi_\varepsilon \Leftrightarrow (H + \varepsilon V)(\psi_0 + \varepsilon \psi_1) = (\lambda_0 + \varepsilon \lambda_1)(\psi_0 + \varepsilon \psi_1) \\
\Leftrightarrow \quad & \varepsilon(H\psi_0 + V\psi_0) + \varepsilon^2 V\psi_1 = \varepsilon(\lambda_0\psi_1 + \lambda_1\psi_0) + \varepsilon^2 \lambda_1 \psi_1 \\
\Leftrightarrow \quad & (H - \lambda_0)\psi_1 = (\lambda_1 - V)\psi_0 + \varepsilon(\lambda_1 - V)\psi_1. 
\end{aligned} \tag{8.8}$$

In order to invert the above equation in $\Sigma$, we *need* according to Proposition 8.1.3 to ensure that the right hand side is orthogonal to $\psi_0$ , and this is therefore a non linear constraint on $\psi_1$. We argue as follows. Given *any* $\psi_1 \in \Sigma$, we define $\lambda_1(\psi_1)$ by the condition

$$\langle (\lambda_1 - V)\psi_0 + \varepsilon(\lambda_1 - V)\psi_1, \psi_0\rangle_{L^2} = 0 \Leftrightarrow \lambda_1(\psi_1) = \frac{\langle V(\psi_0 + \varepsilon\psi_1), \psi_0\rangle_{L^2}}{\langle \psi_0 + \varepsilon\psi_1, \psi_0\rangle_{L^2}}$$

which is well defined for $|\varepsilon| < \varepsilon(\|\psi_1\|_{L^2})$ small enough. We then define

$$F(\psi_1) = (\lambda_1(\psi_1) - V)\psi_0 + \varepsilon(\lambda_1(\psi_1) - V)\psi_1$$

which satisfies

$$\langle F(\psi_1), \psi_0\rangle_{L^2} = 0, \tag{8.9}$$

and (8.8) now becomes

$$(H - \lambda_0)\psi_1 = F(\psi_1) \Leftrightarrow \psi_1 = (H - \lambda_0)^{-1} F(\psi_1) \tag{8.10}$$

where the resolvent is well defined by Proposition 8.1.3 and (8.9). Using (8.9), it is easily shown that (8.10) is solved through Picard fixed point theorem for $|\varepsilon| < \varepsilon_0$ small enough, the details are left to the reader. The key here is the continuity of the resolvent. The conclusions of Proposition 8.1.4 immediately follow. $\square$

The above argument can be extended to any kind of perturbations, including non local, non self adjoint and non linear ones, see for example Exercice 8.1. The key is to reduce the analysis to a Picard fixed point Theorem, the heart of the matter being to understand the continuity of the resolvent in suitable function spaces. Here we used only the $L^2$ bound, this may not be sufficient to study more complicated non linear problems, but a various different kind of estimates can be derived for the resolvent. This is a classical scheme for non linear analysis: derive linear estimates adapted to the structure of the non linear problem.

Let us also say that the above argument *starts the branch of bifurcation* for small $\varepsilon$. One may then ask what happens when we push the branch and let $\varepsilon$ grow: does the first eigenvalue disappear, are other bifurcation branches createds, etc... This kind of question can become very complicated and is very much studied from the physical, mathematical and numerical point of views.

## 8.2 Euler equations and new soliton solutions

We conclude this section by giving two examples attached to the Euler equation where completely different methods apply to produce non linear solitary waves. We shall in particular emphasize that sometimes, non linear ODE's may save the day.

### 8.2.1 The incompressible Euler equation: travelling wave vortices in two dimension

### 8.2.2 The compressible Euler equation: implosion in three dimension

## 8.3 Exercices

**Exercice 8.1** (Non linear bifurcation)**.** Let $H$ be the one dimensional harmonic oscillator with bound state $(\lambda_0 = 2, \psi_0(x) = e^{-\frac{x^2}{2}})$. We aim at solving the non linear equation

$$Hu - \lambda u = u|u|^2.$$

(*i*) Let $u = \varepsilon v$, write the equation for $\varepsilon$.

(*ii*) Prove using a Lyapounov Schmidt bifurcation argument that there exist solutions to the $v$ equation which bifurcate from the ground state $(\lambda_0, \psi_0)$. Compute the law for the deformed non linear eigenvalue.

(*iii*) Can we apply this argument to find the soliton solutions to (7.2)?

# Chapter 9

# Orbital stability of the ground state

We have obtained in chapter 7 the existence of $H^1$ solutions to

$$\left| \begin{array}{l} \Delta Q - Q + Q|Q|^{p-1} = 0 \\ Q \in H^1(\mathbb{R}^d) \end{array} \right. \tag{9.1}$$

in dimension $d \geq 1$ and for all $p > 1$ satisfying (6.2). Every solution to (9.1) induces a solitary wave time periodic solution to $(NLS)$ via the formula

$$u(t,x) = Q(x)e^{it}.$$

A fundamental problem is stability: are these particular solutions stable by perturbation of the initial data for (NLS)? In fact, as can been observed numerically, a generic solution to (9.1) tends to be an *unstable* solution. But the *ground state solution* $Q > 0$ will turn out to be stable in a suitable sense for not too strong nonlinearities and this will a consequence of its variational characterization.

The aim of this chapter is to prove the so called orbital stability of the ground state solitary wave for $s_c < 0$ by following the steps of the seminal proof by Cazenave-Lions [6]. The heart of the proof is Lions' concentration-compactness Lemma, [27], which describes the lack of compactness of the Sobolev embedding $H^1 \subset L^p$, $2 \leq p < 2^*$.

## 9.1 Orbital stability of the solitary wave

We work in this chapter in $\mathbb{R}^d$, $d \geq 1$, and for a nonlinearity $s_c < 0$ ie $1 < p < 1 + \frac{4}{d}$. We let $Q$ be the unique ground state solitary wave given by Theorem 7.2.1.

### 9.1.1 Trivial instabilities

Let $u_0 \in H^1$ and $u \in \mathcal{C}([0, +\infty[; H^1)$ be the corresponding global solution to (6.1) given by Theorem 6.2.1. From the dynamical system point of view, the natural stability statement in the energy space $H^1$ would be the following: for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all $u_0 \in H^1$,

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon) \;\Rightarrow\; \sup_{t \geq 0} \|u(t,x) - Q(x)e^{it}\|_{H^1} < \varepsilon. \tag{9.2}$$

However for (6.1), the symmetry group through scaling and Galilean drifts generates trivial instabilities which violate (9.2).

Scaling instability. For $\lambda > 0$, the solution to (6.1) with data

$$(u_0)_\lambda(x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) \ \text{ is } \ u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} Q(\lambda x) e^{i\lambda^2 t}.$$

We have

$$\|(u_0)_\lambda - Q\|_{H^1} \lesssim |\lambda - 1| \to 0 \ \text{ as } \ \lambda \to 1.$$

But for $t = t_k \stackrel{\text{def}}{=} 2k\pi/\lambda_k^2$ with $\lambda_k^2 \stackrel{\text{def}}{=} (2k)/(2k+1)$,

$$u_{\lambda_k}(t_k, x) = \lambda_k^{\frac{2}{p-1}} Q(\lambda_k x) \quad \text{while} \quad Q(x)e^{it_k} = -Q(x).$$

Hence for all $k$ large enough

$$\sup_{t \geq 0} \|u_{\lambda_k}(t,x) - Q(x)e^{it}\|_{H^1} \geq \|Q\|_{H^1}.$$

Galilean kick. For $\beta \in \mathbb{R}^d$, the solution to (6.1) with data

$$(u_0)_\beta(x) = Q(x)e^{i\beta\cdot x} \ \text{ is } \ u_\beta(t,x) = Q(x - 2\beta t)e^{i\beta\cdot(x-\beta t)}.$$

It satisfies

$$\|(u_0)_\beta - Q\|_{H^1} \lesssim |\beta| \to 0 \ \text{ when } \ \beta \to 0$$

but

$$\forall \beta \in \mathbb{R}^d \setminus \{0\}, \ \sup_{t\geq 0} \|u_\beta(t,x) - Q(x)e^{it}\|_{H^1} \geq \|Q\|_{H^1}$$

because of the decoupling in space of $Q(x)$ et $Q(x - \beta t)$.

### 9.1.2 Orbital stability

The above instabilities simply mean that we shoud not try to control the distance of the solution to the solitary wave picked by the data, but to the *full manifold* of solitary waves generated by the symmetry group.

**Theoreme 9.1.1** (Orbital stability of the ground state, [6])**.** *Let $d \geq 1$ and $1 < p < 1 + \frac{4}{d}\cdot$. Then for all $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for all data $u_0 \in H^1$ with*

$$\|u_0 - Q\|_{H^1} < \delta(\varepsilon),$$

*there exists two functions $\gamma : \mathbb{R}^+ \to \mathbb{R}$ and $x : \mathbb{R}^+ \to \mathbb{R}^d$ such that the solution $u \in \mathcal{C}([0,+\infty[; H^1)$ of (6.1) satisfies:*

$$\sup_{t\geq 0} \|u(t,x) - Q(x - x(t))e^{i\gamma(t)}\|_{H^1} < \varepsilon. \tag{9.3}$$

In other words, a data which is close to the ground state generates a solution which for all time is close to the manifold of solitary waves, and it draws on it a curve through the modulation parameters $(\gamma(t), x(t))$. Theorem 9.1.1 is the starting point of a very active research area with many remaining open problems in connection to asymptotic stability problems (does the solution asymptotically converge as $t \to +\infty$ to a solitary wave?) and stability problems in particular in fluid mechanics.

### 9.1.3 Sharp variational characterization of the ground state

The rest of this chapter is devoted to the proof of Theorem 9.1.1. The fundamental observation is that stability in the energy space does not rely on fine properties of the flow, but simply on a variational characterization of the ground state *based on the conserved invariants of (NLS)*, here mass and energy. Theorem 9.1.1 is indeed a direct consequence of the following:

**Theoreme 9.1.2** (Subcritical characterization of the ground state)**.** *Let $d \geq 1$ and $s_c \stackrel{def}{=} \frac{d}{2} - \frac{2}{p-1} < 0$. Let $Q$ be the ground state profile of Theorem 7.2.1. Let $M > 0$. Then :*

*(i) The minimization problem*

$$I(M) = \inf\left\{E(u) : \quad u \in H^1 \quad \textit{with} \quad \|u\|_{L^2}^2 = M\right\}$$

*where $E(u)$ is the energy functional is attained on the family*

$$Q_{\lambda(M)}(x - x_0)e^{i\gamma_0}, \quad x_0 \in \mathbb{R}^d, \quad \gamma_0 \in \mathbb{R}$$

*where*

$$Q_{\lambda(M)}(x) \stackrel{def}{=} (\lambda(M))^{\frac{2}{p-1}} Q(\lambda(M)x) \quad \textit{avec} \quad \lambda(M) \stackrel{def}{=} \left(\frac{M}{\|Q\|_{L^2}^2}\right)^{-\frac{1}{2s_c}}. \tag{9.4}$$

*(ii) Every minimizing sequence is relatively compact in $H^1$ up to translation and phase shift: let $(u_n)_{n\in\mathbb{N}} \in H^1$ with*

$$\|u_n\|_{L^2}^2 \to M \ \textit{and}\ E(u_n) \to I(M), \tag{9.5}$$

*then there exists $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$, $\gamma \in \mathbb{R}$ and a subsequence $\phi : \mathbb{N} \to \mathbb{N}$ such that:*

$$u_{\phi(n)}(\cdot + x_{\phi(n)})e^{i\gamma} \to Q_{\lambda(M)} \quad \textit{in} \quad H^1. \tag{9.6}$$

*Remark.* The assumption $p < 1 + \frac{4}{d}$ is fundamental for this new characterization of $Q$ which is false for $p \geq 1 + \frac{4}{d}\cdot$. In this last case, the soliton is a saddle point, and it is dynamically unstable by scattering and blow up: any neighborhood of the mass critical and super critical (NLS) ground state contains data which either lead to forward global in time scattering solutions or finite time blow up solutions.

*Proof of Theorem 9.1.1 assuming Theorem 9.1.2.* By contradiction: let $\varepsilon > 0$ and a sequence $(u_n)_{n\in\mathbb{N}}$ of solutions of (6.1) such that

$$\|u_n(0, x) - Q\|_{H^1} \to 0 \quad \text{as} \quad n \to +\infty, \tag{9.7}$$

and there exists $(t_n)_{n\in\mathbb{N}}$ with $t_n \geq 0$ satisfying

$$\forall x_0 \in \mathbb{R}^d, \quad \forall \gamma \in \mathbb{R}, \quad \|u_n(t_n, x) - Q(x - x_0)e^{i\gamma}\|_{H^1} > \varepsilon. \tag{9.8}$$

By (9.7) and the continuity of the energy functional on $H^1$,

$$E(u_n(0, x)) \to E(Q) \quad \text{and} \quad \|u_n(0, x)\|_{L^2}^2 \to \|Q\|_{L^2}^2.$$

Let $w_n(x) = u_n(t_n, x)$, we conclude by *the conservation of mass and energy* that:

$$E(w_n) \to E(Q), \quad \|w_n\|_{L^2}^2 \to \|Q\|_{L^2}^2,$$

and hence by (9.6), we can find $(x_{\phi_n})_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ and $\gamma \in \mathbb{R}$ such that

$$w_{\phi(n)}(\cdot + x_{\phi(n)})e^{i\gamma} \to Q \quad \text{in} \quad H^1$$

which contradicts (9.8). $\square$

## 9.2 Minimization of the energy under a mass constraint

The rest of this chapter is devoted to the proof of Theorem 9.1.2.

### 9.2.1 Computing $I(M)$

We first observe that $I(M)$ is a homoegeneous function of $M$.

**Lemma 9.2.1 (Calcul de $I(M)$).** *There holds*

$$\left|\begin{array}{l}\forall M>0,\quad I(M)=M^{\frac{1-s_c}{|s_c|}}I(1)\\ -\infty<I(1)<0.\end{array}\right. \tag{9.9}$$

*Proof of Lemma 9.2.1.* We first claim.

$$I(M)>-\infty. \tag{9.10}$$

Indeed, from Gagliardo-Nirenberg:

$$\|u\|_{L^{p+1}}\leq C\|\nabla u\|_{L^2}^{\sigma}\|u\|_{L^2}^{1-\sigma}\quad\text{with}\quad -\sigma+\frac{d}{2}=\frac{d}{p+1}.$$

Hence for $u\in H^1$ with $\|u\|_{L^2}^2=M$,

$$E(u)\geq\frac{1}{2}\|\nabla u\|_{L^2}^2-C\|\nabla u\|_{L^2}^{\frac{d(p-1)}{2}}M^{\frac{(p+1)(1-\sigma)}{2}}. \tag{9.11}$$

Since

$$p<1+\frac{4}{d}\Leftrightarrow\frac{d(p-1)}{2}<2,$$

the function $x\mapsto\frac{1}{2}x^2-Cx^{\frac{d(p-1)}{2}}$ is lower bounded on $\mathbb{R}^+$, and (9.10) follows.
We now claim

$$I(M)<0. \tag{9.12}$$

Indeed, let $u\in H^1$ with $\|u\|_{L^2}^2=M$. Fpr $\lambda>0$, let

$$u_\lambda(x)=\lambda^{\frac{d}{2}}u(\lambda x),$$

then

$$\|u_\lambda\|_{L^2}^2=\|u\|_{L^2}^2=M$$

and

$$E(u_\lambda)=\lambda^2\left(\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\,dx-\frac{\lambda^{(p-1)s_c}}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}\,dx\right)<0$$

for $\lambda>0$ small enough.
We now prove (9.9) using a different scaling:

$$u_\lambda(x)=\lambda^{\frac{2}{p-1}}u(\lambda x)$$

which yields

$$\|u_\lambda\|_{L^2}^2=\lambda^{\frac{4}{p-1}-d}\|u\|_{L^2}^2=\lambda^{-2s_c}\|u\|_{L^2}^2$$

and

$$E(u_\lambda)=\lambda^{2(1-s_c)}E(u).$$

Hence:

$$\forall M>0,\quad\forall\lambda>0,\quad I(\lambda^{-2s_c}M)=\lambda^{2(1-s_c)}I(M)$$

and taking $\lambda^{-2s_c}M=1\Leftrightarrow\lambda=M^{\frac{1}{2s_c}}$ yields (9.9). □

### 9.2.2 Classification of minimizers

We assume in this section that the infimum is attained (which will be proved later), and we classify the set of minimizers.

**Lemma 9.2.2** (Euler-Lagrange for minimizers)**.** *Let $u$ be a minimizer for $I$. Then:*

*(i) $|u|$ is a minimizer and*

$$\int |\nabla |u||^2\, dx = \int |\nabla u|^2\, dx. \tag{9.13}$$

*(ii) If $u \geq 0$ then there exists $\mu \in \mathbb{R}$ such that:*

$$\Delta u + u^p = \mu u. \tag{9.14}$$

*(iii) The Lagrange multiplier $\mu$ does not depend on the minimizer and*

$$\mu = \mu(M) > 0.$$

*Proof of Lemma 9.2.2.* The proof is similar to the one of Proposition 7.2.1 : if $u$ is a minimizer, then so is $|u|$ by (7.7), and $E(u) = E(|u|) = I(M)$ implies $\int |\nabla u|^2\, dx = \int |\nabla |u||^2\, dx$, and (i) is proved.
Let $u \geq 0$ be a minimizer and $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then (7.11), (7.12) and (7.13) give (with a slight abuse of notations) :

$$\frac{d}{dt} E(u+th)_{|t=0} = -\int_{\mathbb{R}^d} (\Delta u + u^p) h\, dx \quad \text{and} \quad \frac{d}{dt}\left(\|u+th\|^2_{L^2}\right)_{|t=0} = 2\int_{\mathbb{R}^d} uh\, dx,$$

and as in the proof of Proposition 7.2.1, this yields the existence of $\mu = \mu(u)$ such that $u$ satisfies (9.14). We now claim that $\mu$ does not depend on $u$, which is a consequence of the scale invariance of the minimization problem. Indeed, we multiply (9.14) by $u$ and integrate

$$-\int |\nabla u|^2\, dx + \int u^{p+1}\, dx = \mu \int u^2\, dx = \mu M. \tag{9.15}$$

We then multiply (9.14) by $\frac{d}{2}u + x \cdot \nabla u$. Combining the Pohozaev identity (6.22) with (6.23) for$q = p+1$, we obtain, rembering that $u \geq 0$:

$$0 = -\int |\nabla u|^2\, dx + \int u^p \left(\frac{d}{2}u + x \cdot \nabla u\right) dx = -\int |\nabla u|^2\, dx + \left(\frac{d}{2} - \frac{d}{p+1}\right)\int u^{p+1}\, dx$$

and hence the second relation:

$$\int |\nabla u|^2\, dx = \frac{d(p-1)}{2(p+1)} \int u^{p+1}\, dx.$$

This implies with (9.15):

$$\mu M = \left(1 - \frac{d(p-1)}{2(p+1)}\right) \int u^{p+1}\, dx. \tag{9.16}$$

But by (9.16),

$$\begin{aligned} I(M) &= E(u) = \frac{1}{2}\int |\nabla u|^2\, dx - \frac{1}{p+1}\int u^{p+1}\, dx \\ &= \frac{1}{p+1}\left(\frac{d(p-1)}{4} - 1\right)\int u^{p+1}\, dx = \frac{1}{p+1}\left(\frac{\frac{d(p-1)}{4} - 1}{1 - \frac{d(p-1)}{2(p+1)}}\right)\mu M \end{aligned}$$

and $\mu$ depends only on $M$. We moreover observe that the right hand side is non positive: this is obvious for $d=1$, and for $d\geq 2$:

$$p<1+\frac{4}{d}<\frac{d+2}{d-2}\ \ \text{implies}\ \ \frac{\frac{d(p-1)}{4}-1}{1-\frac{d(p-1)}{2(p+1)}}<0.$$

Since $I(M)<0$, we conclude $\mu>0$. □

We are thus left with classifiying the *positive* $H^1$ solutions to

$$\Delta u+u^p=\mu u,\quad \mu>0.$$

This is a highly non trivial problem which has been solved in the 80's by Gidas, Ni, Nirenberg [15]. This is one of the spectacular success of the analysis of non linear elliptic PDE's from the 80's.

**Theoreme 9.2.1** (Uniqueness of the ground state)**.** *Let $u\in H^1$ be a solution to*

$$\Delta u-u+u^p=0,\quad u\geq 0.$$

*Then there exists $x_0\in\mathbb{R}^d$ such that $u(x-x_0)$ has spherical symmetry.*

Let us stress that the link between positivity and symmetry is at first hand totally unclear, and the proof relies on a very clever use of the maximum principle for the Laplace operator and the *moving hyperplane* method which goes beyond the scope of these notes. We may now complete the classification of minimizers.

**Proposition 9.2.1** (Classification of minimizers)**.** *Let $u$ be a minimizer of*

$$I(M)=\inf\big\{E(u):\quad u\in H^1\quad \textit{with}\quad \|u\|_{L^2}=M\big\}\cdot$$

*Then there exists $(\gamma_0,x_0)\in\mathbb{R}\times\mathbb{R}^d$ such that*

$$u(x)=Q_{\lambda(M)}(x-x_0)e^{i\gamma_0}$$

*where $Q_{\lambda(M)}$ is given by* (9.4)*.*

*Proof of Proposition 9.2.2.* Let u be a minimizer, then so is $v=|u|\geq 0$. By Lemma 9.2.2, $v$ is a non trivial (since $I(M)<0$) solution to

$$\Delta v+v^p=\mu v,\quad v\in H^1,\quad v\geq 0$$

with $\mu=\mu(M)>0$. Then

$$w=\frac{1}{\lambda^{\frac{2}{p-1}}}v\Big(\frac{x}{\lambda}\Big)\ \ \text{with}\ \ \lambda=\sqrt{\mu} \tag{9.17}$$

satisfies

$$\Delta w-w+w^p=0,\quad w\in H^1,\quad w\geq 0.$$

Hence the combination of Theorem 7.2.1 and Theorem 9.2.1 ensure that $w=Q(x-x_0)$ for some $x_0\in\mathbb{R}^d$. Coming back to (9.17), we obtain

$$\|Q\|_{L^2}^2=\|w\|_{L^2}^2=\lambda^{2s_c}\|v\|_{L^2}^2=\lambda^{2s_c}M.$$

Since $Q$ does not vanish, we conclude by (9.13) and Lemma 7.2.2 that there exists $\gamma\in\mathbb{R}$ and $x_1\in\mathbb{R}^d$ such that

$$u=|u|e^{i\gamma}=Q_{\lambda(M)}(x-x_1)e^{i\gamma}.$$

□

## 9.3 The $H^1$ relative compactness of minimizing sequences

We now turn to the heart of the description of minimizing sequences which will allow us to conclude the proof of Theorem 9.1.2.

### 9.3.1 The lack of compactness of the Sobolev injection

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for $I(M)$:

$$\|u_n\|_{L^2}^2 = M, \quad E(u_n) \to I(M).$$

Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1$ from (9.11). To show that one can extract a non trivial weak limit which attains the infimun, we must establish strong convergence in $L^{p+1}\cap L^2$, which is of course completely false for a generic bounded $H^1$ sequence. To overcome this difficulty, we introduce the *profile decomposition* of the sequence $u_n$ as introduced in [14] in the continuation of [27] which completely describes the lack of compactness of the Sobolev embedding.

Let us start with a definition.

**Definition** (Limiting weak set)**.** Let $\mathbf{v} = (v_n)_{n\geq 1}$ be a bounded sequence of $H^1$ functions. We let $\mathcal{V}(\mathbf{v})$ be the set of all possible $H^1$ weak limits extracted from $(v_n)_{n\geq 1}$ and its translates:

$$V \in \mathcal{V}(\mathbf{v}) \Leftrightarrow v_{\phi(n)}(\cdot + x_n) \rightharpoonup V \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty.$$

This is a bounded subset of $H^1$ and we denote

$$\eta(\mathbf{v}) = \sup_{V\in\mathcal{V}(\mathbf{v})} \|V\|_{H^1}.$$

We may now state the profile decomposition property.

**Proposition** (Profile decomposition)**.** *Let $d\geq 1$. Let $\mathbf{v} = (v_n)_{n\geq 1}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then there exists a subsequence still denoted $(v_n)_{n\geq 1}$, a family $(\mathbf{x}^j)_{j\geq 1}$ of sequences $(x_n^j)_{n\geq 1}$ of points of $\mathbb{R}^d$, a sequence of profiles $(V^j)_{j\geq 1}$ bounded in $H^1$ and a family $(\mathbf{v}^j)_{j\geq 1}$ of sequences of corrections $v_n^j \in H^1$ such that the following holds:*
(i) Separation*: for $k\neq j$,*

$$\lim_{n\to+\infty} |x_n^k - x_n^j| = +\infty. \tag{9.18}$$

(ii) Decomposition in $H^1$*: there holds the decomposition*

$$v_n = \sum_{j=1}^{\ell} V^j(\cdot - x_n^j) + v_n^\ell \tag{9.19}$$

*with for every fixed $\ell\geq 1$:*

$$\left| \begin{array}{l} \|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1) \\ \|\nabla v_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1) \end{array} \right. \tag{9.20}$$

*and the asymptotic vanishing in $\ell$:*

$$\lim_{\ell\to+\infty} \eta(\mathbf{v}^\ell) = 0. \tag{9.21}$$

(iii) Uniform splitting of the potential energy*: let*

$$2 < p < 2^* \tag{9.22}$$

*then*

$$\lim_{\ell \to +\infty} \limsup_{n \to \infty} \|v_n^\ell\|_{L^p} = 0 \tag{9.23}$$

*and there holds the no loss asymptotic splitting of the potential energy:*

$$\left| \begin{array}{l} \|v_n\|_{L^p}^p = \sum_{j=1}^\ell \|V^j\|_{L^p}^p + \varepsilon_n^\ell \\ \lim_{\ell \to +\infty} \limsup_{n \to +\infty} \varepsilon_n^\ell = 0. \end{array} \right. \tag{9.24}$$

**Remark 9.3.1.** *Beware of the fact that* (9.21) *is very different from* $\lim_{\ell \to +\infty} \limsup_{n \to \infty} \|v_n^\ell\|_{H^1} = 0$ *which has* no reason to hold in general *as can be seen on the example of the vanishing sequence*

$$v_n = \frac{1}{n^{\frac{d}{2}}} \phi\left(\frac{x}{n}\right), \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

*It however holds in* $L^p$ *for* $2 < p < 2^*$, (9.23)*, this is why a compactess process is at work.*

In other words, up to an error which can be made arbitrarily small in in $L^p$, smallness being indexed by the parameter $\ell$, the sequence $v_n$ constists of $\ell$ bubbles which move strictly away from each other according to (9.18). The decomposition can be iterated $\ell \to +\infty$ with smaller and smaller errors in $L^p$ and in the $\eta(\mathbf{v}^\ell)$ sense which provides an asymptotic no loss estimate for the potential energy (9.23), (9.24) which is contained in the bubbles only.

*Proof of Proposition 9.3.1.* We closely follow [19]. We fix once and for all $\mathbf{v} = (v_n)_{n \geq 1}$ a bounded sequence in $H^1$.

**step 1** Induction and $H^1$ bounds. We construct by induction on $j$ a sequence $V^j \in \mathcal{V}(\mathbf{v})$, a family $(x^j)_{j \geq 1}$ of sequence of $\mathbb{R}^d$ such that (9.18) holds and a family $(v^j)_{j \geq 1}$ of sequences of error $v_n^j \in H^1$ such that, up to a subsequence, (9.19) holds with the uniform bound (9.21).

<u>$\ell = 1$</u>. If $\eta(\mathbf{v}) = 0$, we can take $V^j = 0$ for all $j$. Otherwise, we pick a non trivial profile $V^1 \in \mathcal{V}(\mathbf{v})$ such that

$$\|V^1\|_{H^1} \geq \frac{1}{2}\eta(\mathbf{v}) > 0.$$

Then by definition, there exists $\mathbf{x}^1 = (x_n^1)_{n \geq 1}$ such that up to extraction of a subsequence:

$$v_n(\cdot + x_n^1) \rightharpoonup V^1 \text{ in } H^1$$

and we set

$$v_n^1 = v_n - V^1(\cdot - x_n^1).$$

Since $v_n^1(\cdot + x_n^1) \rightharpoonup 0$ in $H^1$, we have as $n \to +\infty$ and by translation invariance of the Lebesgue measure in $\mathbb{R}^d$:

$$\begin{aligned} \|v_n\|_{L^2}^2 &= \|v_n^1 + V^1(\cdot - x_n^1)\|_{L^2}^2 = \|v_n^1\|_{L^2}^2 + 2\Re \left\langle v_n^1, V^1(\cdot - x_n^1)\right\rangle_{L^2} + \|V^1(\cdot - x_n^1)\|_{L^2}^2 \\ &= \|v_n^1\|_{L^2}^2 + \|V^1\|_{L^2}^2 + 2\Re \left\langle v_n^1(\cdot + x_n^1), V^1 \right\rangle_{L^2} = \|v_n^1\|_{L^2}^2 + \|V^1\|_{L^2}^2 + o_{n \to +\infty}(1) \end{aligned}$$

and similarly

$$\|\nabla v_n\|_{L^2}^2 = \|\nabla v_n^1\|_{L^2}^2 + \|\nabla V^1\|_{L^2}^2 + o_{n \to +\infty}(1).$$

This implies the bounds

$$\limsup_{n\to+\infty} \|v_n^1\|_{H^1}^2 \leq \limsup_{n\to+\infty} \|v_n\|_{H^1}^2.$$

$\underline{\ell = 2}$. We now write

$$v_n = V^1(\cdot - x_n^1) + v_n^1,$$

replace $\mathbf{v}$ by $\mathbf{v^1} = (v_n^1)_{n\geq 1}$ which is a bounded sequence in $H^1$, and iterate the process. If $\eta(\mathbf{v}^1) = 0$, we take $V^j = 0$ for all $j \geq 2$. Otherwise, we extract $V^2 \neq 0$, $\mathbf{x}^2$, $\mathbf{v}^2$ as above. The fundamental observation is that necessarily the sequences $x^1$, $x^2$ decouple ie

$$\lim_{n\to+\infty} |x_n^1 - x_n^2| = +\infty.$$

Indeed, by contradiction, we could otherwise extract up to a subsequence

$$x_n^1 - x_n^2 \to x_0 \in \mathbb{R}^d \quad \text{as} \quad n \to +\infty, \tag{9.25}$$

but then since by construction

$$\left| \begin{array}{l} v_n^1(\cdot + x_n^2) \rightharpoonup V^2 \quad \text{in} \quad H^1 \\ v_n^1(\cdot + x_n^1) \rightharpoonup 0 \quad \text{in} \quad H^1 \end{array} \right.$$

the relation

$$v_n^1(\cdot + x_n^2)) = v_n^1(\cdot + (x_n^2 - x_n^1) + x_n^1))$$

with (9.25) implies

$$V^2 = 0$$

which is a contradiction. Finally, we observe that by construction

$$\left| \begin{array}{l} v_n^1(\cdot + x_n^1) \rightharpoonup 0 \quad \text{in} \quad H^1 \\ v_n^2(\cdot + x_n^2) \rightharpoonup 0 \quad \text{in} \quad H^1 \end{array} \right.$$

and

$$v_n^1 = V^2(\cdot - x_n^2) + v_n^2 \Rightarrow v_n^1(\cdot + x_n^1) = V^2(\cdot + x_n^1 - x_n^2) + v_n^2(\cdot + x_n^1).$$

Since $|x_n^1 - x_n^2| \to +\infty$,

$$V^2(\cdot + x_n^1 - x_n^2) \rightharpoonup 0 \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty$$

and hence

$$v_n^2(\cdot + x_n^1) \rightharpoonup 0 \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty.$$

This implies

$$v_n = S_n^2 + v_n^2, \quad S_n^2 = \sum_{j=1}^{2} V^j(\cdot - x_n^j).$$

We compute the norms:

$$\left| \begin{array}{l} \|v_n\|_{L^2}^2 = \|S_n^2 + v_n^2\|_{L^2}^2 = \|S_n^2\|_{L^2}^2 + 2\Re\left\langle S_n^2, v_n^2\right\rangle_{L^2} + \|S_n^2\|_{L^2}^2 \\ \nabla v_n\|_{L^2}^2 = \|\nabla S_n^2 + \nabla v_n^2\|_{L^2}^2 = \|\nabla S_n^2\|_{L^2}^2 + 2\Re\left\langle \nabla S_n^2, \nabla v_n^2\right\rangle_{L^2} + \|\nabla S_n^2\|_{L^2}^2 \end{array} \right.$$

The cross product vanishes at $n \to +\infty$ using

$$v_n^2(\cdot + x_n^j) \rightharpoonup 0 \quad \text{in} \quad H^1 \quad \text{as} \quad n \to +\infty, \quad j = 1, 2$$

which ensures

$$\left|\begin{array}{l}\Re\left\langle V_j(\cdot - x_n^j, v_n^2\right\rangle_{L^2} = \Re\left\langle V_j, v_n^2(\cdot + x_n^j)\right\rangle_{L^2} \to 0 \ \text{ as } \ n\to+\infty \\ \Re\left\langle \nabla V_j(\cdot - x_n^j, \nabla v_n^2\right\rangle_{L^2} = \Re\left\langle \nabla V_j, \nabla v_n^2(\cdot + x_n^j)\right\rangle_{L^2} = 0.\end{array}\right.$$

On the other hand, the decoupling $\lim_{n\to+\infty}|x_n^1 - x_n^2| = +\infty$ ensures

$$\begin{aligned}\|S_n^2\|_{L^2}^2 &= \|V^1(\cdot - x_n^1) + V^2(\cdot - x_n^2)\|_{L^2}^2 = \|V^1(\cdot - x_n^1)\|_{L^2}^2 + \|V^2(\cdot - x_n^2)\|_{L^2} + o_{n\to+\infty}(1)\\ &= \|V^1\|_{L^2} + \|V^2\|_{L^2}^2 + o_{n\to+\infty(1)}\end{aligned}$$

and similarily for the Dirichlet energy:

$$\|\nabla S_n^2\|_{L^2}^2 = \|\nabla V^1\|_{L^2} + \|\nabla V^2\|_{L^2}^2 + o_{n\to+\infty(1)}.$$

Hence

$$\left|\begin{array}{l}\|v_n\|_{L^2}^2 = \|V^1\|_{L^2}^2 + \|V^2\|_{L^2}^2 + \|v_n^2\|_{L^2}^2 + o_{n\to+\infty}(1)\\ \|\nabla v_n\|_{L^2}^2 = \|\nabla V^1\|_{L^2}^2 + \|\nabla V^2\|_{L^2}^2 + \|\nabla v_n^2\|_{L^2}^2 + o_{n\to+\infty}(1).\end{array}\right.$$

<u>Induction on $\ell$</u>. We now argue by induction on $\ell$ and use a diagonal extraction argument to construct $V^\ell$, $x^\ell$, $\mathbf{v}^\ell$ such that up to a subsequence, for all $\ell\geq 1$,

$$\left|\begin{array}{l} v_n = S_n^\ell + v_n^\ell\\ S_n^\ell = \sum_{j=1}^\ell V^j(\cdot - x_n^j),\end{array}\right. \tag{9.26}$$

the separation (9.18) holds, the weak limit

$$\forall 1\leq j\leq \ell, \quad v_n^\ell(\cdot + x_n^j) \rightharpoonup 0 \ \text{ in } \ H^1 \ \text{ as } \ n\to+\infty \tag{9.27}$$

holds and by construction

$$\eta(\mathbf{v^j}) \leq 2\|V^{j-1}\|_{H^1}, \quad j\geq 2. \tag{9.28}$$

Fix $\ell\geq 1$, then using (9.18) yields the asymptotic orthogonality:

$$\left|\begin{array}{l}\|S_n^\ell\|_{L^2}^2 = \|\sum_{j=1}^\ell V^j(\cdot - x_n^j)\|_{L^2}^2 \to \sum_{j=1}^\ell \|V^j\|_{L^2}^2 \ \text{ as } \ n\to+\infty\\ \|\nabla S_n^\ell\|_{L^2}^2 \to \sum_{j=1}^\ell \|\nabla V^j\|_{L^2}^2 \ \text{ as } \ n\to+\infty.\end{array}\right. \tag{9.29}$$

We now develop the scalar product and use (9.27) to compute:

$$\|v_n\|_{L^2}^2 = \|S_n^\ell + v_n^\ell\|_{L^2}^2 = \|S_n^\ell\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1) = \sum_{j=1}^\ell \|V^j\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1)$$

and similarily for the Dirichelt energy, and (9.20) is proved. This implies letting $n\to+\infty$:

$$C \geq \lim_{\sup n\to+\infty} \|v_n\|_{H^1}^2 \geq \sum_{j=1}^\ell \|V^j\|_{H^1}^2$$

and hence since this is true for all $j$:

$$\sum_{j=1}^{+\infty} \|V^j\|_{H^1}^2 < +\infty. \tag{9.30}$$

This implies recalling (9.28):

$$\eta(\mathbf{v^j}) \le 2\|V^{j-1}\|_{H^1} \to 0 \ \text{ as } \ j \to +\infty$$

and (9.21) is proved.

**step 2** $L^p$ bounds. We now claim the asymptotic splitting of the kinetic energy (9.23), (9.24) which follows from (9.21) and Sobolev embeddings. First observe from (9.20) the uniform bound

$$\forall \ell \ge 1, \quad \limsup_{n\to+\infty} \|v_n^\ell\|_{H^1} \le \limsup_{n\to+\infty} \|v_n\|_{H^1} \le C \tag{9.31}$$

independent of $\ell$. Let us fix once and for all $\chi \in \mathcal{S}(\mathbb{R}^d)$ with

$$\widehat{\chi}(\xi) = \left| \begin{array}{ll} 1 & \text{for } \ |\xi| \le 1 \\ 0 & \text{for } \ |\xi| \ge 2 \end{array} \right. , \quad |\widehat{\chi}| \le 1,$$

and given $R > 0$, let

$$\chi_R(x) = R^d \chi(Rx). \tag{9.32}$$

Then

$$\widehat{\chi_R}(\xi) = \widehat{\chi}\left(\frac{\xi}{R}\right) = \left| \begin{array}{ll} 1 & \text{for } \ |\xi| \le R \\ 0 & \text{for } \ |\xi| \ge 2R \end{array} \right.$$

We now split in low and high frequencies

$$\widehat{v_n^\ell} = \widehat{v_n^\ell}\widehat{\chi_R} + \widehat{v_n^\ell}(1 - \widehat{\chi_R}) \Leftrightarrow v_n^\ell = \chi_R \star v_n^\ell + (\delta - \chi_R) \star v_n^\ell$$

where $\delta$ is the Dirac mass at the origin. High frequencies are estimated using the homegeneous Sobolev embbeding: let

$$-s + \frac{d}{2} = \frac{d}{p}, \quad 0 < s < 1$$

from (9.22), then

$$\begin{aligned} \|(\delta - \chi_R) \star v_n^\ell\|_{L^p}^2 &\lesssim \|(\delta - \chi_R) \star v_n^\ell\|_{\dot{H}^s}^2 \le \int_{|\xi|\ge R} |\xi|^{2s} |\widehat{v_n^\ell}|^2(\xi) d\xi = \int_{|x|\ge R} \frac{|\xi|^2|}{|\xi|^{2(1-s)}} |\widehat{v_n^\ell}|^2(\xi) d\xi \\ &\lesssim \frac{\|v_n^\ell\|_{H^1}^2}{R^{2(1-s)}}. \end{aligned}$$

and hence using (9.31)

$$\forall \ell \ge 1, \quad \limsup_{n\to+\infty} \|(\delta - \chi_R) \star v_n^\ell\|_{L^p}^2 \le \frac{C}{R^{2(1-s)}}.$$

Given $\varepsilon > 0$, we may therefore find $R = R(\varepsilon)$ such that

$$\forall \ell \ge 1, \quad \limsup_{n\to+\infty} \|(\delta - \chi_R) \star v_n^\ell\|_{L^p}^2 < \varepsilon.$$

This $R = R(\varepsilon)$ being now fixed, we estimate the low frequency part using Hölder and Young:

$$\|\chi_R \star v_n^\ell\|_{L^p} \lesssim \|\chi_R \star v_n^\ell\|_{L^2}^{\frac{2}{p}} \|\chi_R \star v_n^\ell\|_{L^\infty}^{1-\frac{2}{p}} \lesssim \|\chi_R\|_{L^1} \|v_n^\ell\|_{L^2} \|\chi_R \star v_n^\ell\|_{L^\infty}^{1-\frac{2}{p}} \lesssim \|v_n^\ell\|_{L^2} \|\chi_R \star v_n^\ell\|_{L^\infty}^{1-\frac{2}{p}}$$

where used from (9.32):

$$\forall R > 0, \quad \|\chi_R\|_{L^1} = \|\chi\|_{L^1}.$$

Hence from (9.31):

$$\limsup_{n\to+\infty} \|\chi_R \star v_n^\ell\|_{L^p} \le C \limsup_{n\to+\infty} \|\chi_R \star v_n^\ell\|_{L^\infty}^{1-\frac{2}{p}}.$$

We now observe recalling the Definition 9.3.1 of the limiting weak set:

$$\limsup_{n\to+\infty} \|\chi_R \star v_n^\ell\|_{L^\infty} = \sup_{(x_n)_{n\ge 1}} \limsup_{n\to+\infty} \|(\chi_R \star v_n^\ell)(x_n)\|_{L^\infty} = \sup_{(x_n)_{n\ge 1}} \limsup_{n\to+\infty} \left|\int_{\mathbb{R}^d} \chi_R(x_n - y) v_n(y) dy\right|$$

$$= \sup_{(x_n)_{n\ge 1}} \limsup_{n\to+\infty} \left|\int_{\mathbb{R}^d} \chi_R(-y) v_n(x_n + y) dy\right| \le \sup_{V \in \mathcal{V}(\mathbf{v}^\ell)} \left|\int_{\mathbb{R}^d} \chi_R(-y) V(y) dy\right|.$$

From Hölder,

$$\left|\int_{\mathbb{R}^d} \chi_R(-y) V(y) dy\right| \le \|\chi_R\|_{L^2} \|V\|_{L^2}$$

and hence the bound:

$$\limsup_{n\to+\infty} \|\chi_R \star v_n^\ell\|_{L^\infty} \le C(R) \eta(v^\ell).$$

$R = R(\varepsilon)$ has been fixed, so we now let $\ell \to +\infty$ and (9.21) yields (9.23). We are now in position to conclude the proof of (9.24). Fix $\ell \ge 1$ and recall (9.26):

$$v_n = S_n^\ell + v_n^\ell.$$

We first estimate from (9.29), (9.30) and Sobolev:

$$\forall \ell \ge 1, \quad \limsup_{n\to+\infty} \|S_n^\ell\|_{L^p}^2 \le C_p \limsup_{n\to+\infty} \|S_n^\ell\|_{H^1}^2 < C \tag{9.33}$$

with constant $C > 1$ independent of $\ell$. We now use the homogeneity estimate

$$\left|\left|\sum_{j=1}^{\ell} a_j\right|^p - \sum_{j=1}^{\ell} |a_j|^p\right| \le C_{p,\ell} \sum_{j\ne q} |a_j||a_k|^{p-1} \tag{9.34}$$

to first estimate with Hölder:

$$\begin{aligned} \left|\|S_n^\ell + v_n^\ell\|_{L^p}^p - \|S_n^\ell\|_{L^p}^p - \|v_n\|_{L^p}^p\right| &\le C_p \int_{\mathbb{R}^d} \left[|S_n^\ell||v_n^\ell|^{p-1} + |v_n^\ell||S_n^\ell|^{p-1}\right] dx \\ &\le C_p \left(\|v_n^\ell\|_{L^p} \|S_n^\ell\|_{L^p}^{p-1} + \|v_n^\ell\|_{L^p}^{p-1} \|S_n^\ell\|_{L^p}\right). \end{aligned} \tag{9.35}$$

We now estimate using (9.34), (9.18) and an elementary density argument:

$$\|S_n^\ell\|_{L^p}^p = \sum_{j=1}^{\ell} \|V^j\|_{L^p}^p + o_{n\to+\infty}(1).$$

Let

$$\varepsilon_n^\ell = \|v_n\|_{L^p}^p - \sum_{j=1}^{\ell} \|V^j\|_{L^p}^p,$$

we conclude using (9.33), (9.35), (9.23) that

$$\forall \ell \ge 1, \quad \limsup_{n\to+\infty} |\varepsilon_n^\ell| \le C_p \limsup_{n\to+\infty} \|v_n^\ell\|_{L^p}$$

and letting $\ell \to +\infty$ and using (9.23) concludes the proof of (9.24).

□

The profile decomposition is the generalization of P.L Lions' concentration compactness principle whihch arose for the very first time at the end of the 1970's in non compact geometric problems and led to the resolution of a series of classical variational problems in mathematical physics, [27]. More recently, a new set of applications arose for the study of nonlinear dispersive equations and the classification of *minimal elements* in the seminal works of Kenig, Merle [21], which have designes a revolutionary route map for the study of global existence and scattering for non linear dispersive PDE's.

### 9.3.2 Compactness of minimizing sequences

Let us show how Proposition 9.3.1 allows us to conclude the proof of Theorem 9.1.2.

Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence. Let us consider up to a subsequence the profile decomposition of Proposition 9.3.1. The key is to show that for a minimizing sequence, the profile decomposition must be trivial

$$V^j = 0 \ \text{ for } \ j \geq 2. \tag{9.36}$$

Indeed, by Proposition 9.3.1,

$$\begin{aligned} E(u_n) &= \frac{1}{2}\|\nabla v_n\|_{L^2}^2 - \frac{1}{p+1}\|u_n\|_{L^{p+1}}^{p+1} = \frac{1}{2}\sum_{j=1}^{\ell}\|\nabla V^j\|_{L^2}^2 + \|\nabla v_n^\ell\|_{L^2}^2 - \frac{1}{p+1}\sum_{j=1}^{\ell}\|V^j\|_{L^{p+1}}^{p+1} - \varepsilon_{n,\ell} \\ &= \sum_{j=1}^{\ell} E(V^j) + \|\nabla v_n^\ell\|_{L^2}^2 + \varepsilon_{n,\ell} \geq \sum_{j=1}^{\ell} E(V^j) - \varepsilon_{n,\ell} \end{aligned}$$

with

$$\lim_{\ell\to+\infty}\limsup_{n\to+\infty}\varepsilon_{n,\ell} = 0.$$

Letting $n \to +\infty$ and then $\ell \to +\infty$ yields

$$I(M) \geq \sum_{j=1}^{+\infty} E(V^j).$$

Moreover,

$$M = \|v_n\|_{L^2}^2 = \sum_{j=1}^{\ell}\|V^j\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1)$$

and hence letting $n \to +\infty$ and then $\ell \to +\infty$ yields

$$M \geq \sum_{j=1}^{+\infty}\|V^j\|_{L^2}^2.$$

Let

$$M_j = \|V^j\|_{L^2}^2 = \alpha_j M, \ \ 0 \leq \alpha_j \leq 1,$$

and recall (9.9):

$$I(M) = M^\beta, \ \ \beta = \frac{1+|s_c|}{|s_c|} > 1,$$

we obtain:

$$I(M) = I(1)M^\beta \geq \sum_{j=1}^{+\infty} E(V^j) \geq \sum_{j=1}^{+\infty} I(M_j) = I(1)M^\beta \sum_{j=1}^{+\infty} \alpha_j^\beta$$

and hence since $I(1) < 0$:

$$\sum_{j=1}^{+\infty} \alpha_j^\beta \geq 1.$$

On the other hand,

$$M \geq \sum_{j=1}^{+\infty} \|V^j\|_{L^2}^2 = M \sum_{j=1}^{+\infty} \alpha_j$$

and hence the constraints

$$\left| \begin{array}{l} \sum_{j=1}^{+\infty} \alpha_j \leq 1 \\ \sum_{j=1}^{+\infty} \alpha_j^\beta \geq 1 \\ 0 \leq \alpha_j \leq 1 \end{array} \right.$$

which since $\beta > 1$ forces $\alpha_1 = 1$, $\alpha_j = 0$ for $j \geq 2$, and (9.36) is proved. We conclude:

$$\left| \begin{array}{l} I(M) \geq E(V^1) \\ \|V^1\|_{L^2}^2 = \alpha_1 M = M \end{array} \right.$$

which since $I(M)$ is the infimum forces

$$E(V^1) = I(M),$$

and $V^1$ attains the infimum. We have shown

$$\|u_n\|_{L^2}^2 \to \|V^1\|_{L^2}^2$$

and by strong $L^{p+1}$ convergence since $V^1$ attains the infimum

$$\|\nabla u_n\|_{L^2}^2 \to \|\nabla V^1\|_{L^2}^2$$

and hence

$$u_n - V^1(\cdot - x_n^1) \to 0 \quad \text{in} \quad H^1.$$

We conclude using Proposition 9.2.1 that there exists $(\gamma_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ such that

$$V^1 = Q_{\lambda(M)}(x - x_0)e^{i\gamma_0},$$

which concludes the proof of (9.6).

## 9.4 Exercices

**Exercice 9.1** (Ground state of a gaseous star)**.** We work in $\mathbb{R}^3$. To every positive function $u : \mathbb{R}^3 \to \mathbb{R}^+$, we associate its Poisson field

$$E_u \stackrel{\text{def}}{=} \nabla \phi_u \quad \text{with} \quad \phi_u \stackrel{\text{def}}{=} -\frac{1}{4\pi|x|} \star u.$$

The potential $\phi_u$ is a solution to

$$\Delta \phi_u = u. \tag{9.37}$$

$(i)$ Show that

$$|E_u(x)| \lesssim \frac{1}{|x|^2} \star |u|.$$

Prove that

$$\|E_u\|_{L^2} \lesssim \|u\|_{L^2}^{\frac{1}{3}} \|u\|_{L^1}^{\frac{2}{3}}.$$

$(ii)$ Compute $\widehat{E_u}$ in terms of $\widehat{u}$. Prove

$$\|E_u\|_{H^1} \lesssim \|u\|_{L^2} + \|u\|_{L^1}.$$

$(iii)$ Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1 \cap L^2$ such that

$$u_n \rightharpoonup u \ \text{ dans } \ L^2.$$

Show using Plancherel that

$$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^3), \quad \int E_{u_n} \overline{\phi}\, dx \longrightarrow \int E_u \overline{\phi}\, dx.$$

Prove that

$$E_{u_n} \rightharpoonup E_u \ \text{ dans } \ L^2.$$

$(iv)$ We assume that $u$ has spherical symmetry. Show the representation formula

$$E_u(r) = \phi_u'(r) e_r = \left( \frac{1}{r^2} \int_0^r \tau^2 u(\tau) d\tau \right) \frac{x}{|x|}.$$

Show that

$$\forall R > 0, \quad \int_{|x|\geq R} |E_u|^2 dx \lesssim \frac{\|u\|_{L^1}^2}{R}.$$

$(v)$ Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^1 \cap L^2$ of radially symmetric positive functions. Show that we can extract $(u_{\varphi(n)})_{n\in\mathbb{N}}$ such that

$$u_{\varphi(n)} \rightharpoonup u \ \text{ in } \ L^2$$

and

$$E_{u_{\varphi(n)}} \to E_u \ \text{ in } \ L^2.$$

$(vi)$ Let $M > 0$ and

$$A(M) = \left\{ u : \mathbb{R}^3 \mapsto \mathbb{R}^+ \ \text{ with } \ u \in L^2(\mathbb{R}^3) \ \text{ et } \ \int_{\mathbb{R}^3} u\, dx = M \right\}.$$

Let

$$I(M) = \inf_{u \in A(M)} \left[ \int_{\mathbb{R}^3} |u|^2\, dx - \int_{\mathbb{R}^3} |E_u|^2\, dx \right].$$

Show that

$$-\infty < I(M) < 0.$$

$(vii)$ Compute $I(M)$ in terms of $M$ and $I(1)$.

(*viii*) Let $A_{rad}(M)$ be the set of radially symmetric elements $u \in A(M)$. Let

$$I_{rad}(M) = \inf_{u \in A_{rad}(M)} \left[ \int_{\mathbb{R}^3} |u|^2 \, dx - \int_{\mathbb{R}^3} |E_u|^2 \, dx \right].$$

Show that $I_{rad}(M)$ is attained.

**Exercice 9.2** (Kinetic model of stars)**.** A galaxy is a cluster of $10^{15}$ stars. A statistic description is given by the distribution $f(x,v)$ which is the density of stars which have the speed $v \in \mathbb{R}^3$ at the point $x \in \mathbb{R}^3$. The total number of stars at $x \in \mathbb{R}^3$ is therefore

$$\rho_f(x) = \int_{v \in \mathbb{R}^3} f(x,v) \, dv.,$$

and the total number of stars is

$$\|f\|_{L^1(\mathbb{R}^6)} = \int_{\mathbb{R}^6} f(x,v) \, dx \, dv = \int_{\mathbb{R}^3} \rho_f(x) \, dx.$$

The total kinetic energy of the galaxy is

$$E_{cin}(f) = \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f(x,v) \, dx \, dv.$$

Last, stars are submitted only to the gravitational force, and the total potential energy is

$$E_{pot}(f) = \int_{\mathbb{R}^3} |\nabla \phi_f(x)|^2 \, dx \quad \text{où} \quad \phi_f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_f(y)}{|x-y|} \, dy.$$

Given $M_1, M_2 > 0$, we consider the minimization problem:

$$I(M_1, M_2) = \inf_{f \in \mathcal{A}(M_1.M_2)} E(f)$$

which defines a stable galazy, where

$$A(M_1.M_2) = \left\{ f(x,v) \geq 0, \ \ \|f\|_{L^1(\mathbb{R}^6)} = M_1, \ \ \|f\|_{L^2(\mathbb{R}^6)} = M_2 \right\}$$

and

$$E(f) = \frac{1}{2} \int_{\mathbb{R}^6} |v|^2 f \, dx \, dv - \int_{\mathbb{R}^3} |\nabla \phi_f(x)|^2 \, dx.$$

(*i*) Let $x \in \mathbb{R}^3$. By splitting $|v| \leq R$ et $|v| \geq R$, show that

$$|\rho_f(x)| \lesssim R^{\frac{3}{2}} \left( \int_{\mathbb{R}^3} f^2(x,v) \, dv \right)^{\frac{1}{2}} + \frac{1}{R^2} \int_{\mathbb{R}^3} |v|^2 f(x,v) \, dv.$$

(*ii*) Conclude by optimizing on $R$ that

$$\forall x \in \mathbb{R}^3, \ \ |\rho_f(x)| \lesssim \left( \int_{\mathbb{R}^3} |v|^2 f(x,v) \, dv \right)^{\frac{3}{7}} \left( \int_{\mathbb{R}^3} f^2(x,v) \, dv \right)^{\frac{2}{7}}.$$

(*iii*) Prove using Hölder:

$$\|\rho_f\|_{L^{\frac{7}{5}}(\mathbb{R}^3)} \lesssim \||v|^2 f\|_{L^1(\mathbb{R}^6)}^{\frac{3}{7}} \|f\|_{L^2(\mathbb{R}^6)}^{\frac{4}{7}}.$$

$(iv)$ Prove using Hölder:

$$\|\rho_f\|^2_{L^{\frac{6}{5}}(\mathbb{R}^3)} \lesssim \||v|^2 f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{5}{6}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^6)}.$$

$(v)$ Show that

$$|\nabla\phi_f(x)| \lesssim \frac{1}{|x|^2} \star \rho_f$$

and obtain the interpolation estimate

$$\int |\nabla\phi_f(x)|^2 \, dx \lesssim \||v|^2 f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{5}{6}}_{L^1(\mathbb{R}^6)} \|f\|^{\frac{2}{3}}_{L^2(\mathbb{R}^6)}.$$

$(vi)$ Show that

$$I(M_1, M_2) > -\infty.$$

$(vii)$ Using the scaling

$$f_\lambda(x,v) = f\left(\frac{x}{\lambda}, \lambda v\right), \quad \lambda > 0$$

show that

$$I(M_1, M_2) < 0.$$

$(viii)$ Using the scaling

$$f_{\lambda,\mu}(x,v) = \frac{\mu}{\lambda^2} f\left(\frac{x}{\lambda}, \mu v\right), \quad \lambda, \mu > 0,$$

show that

$$I(M_1, M_2) = M_1^{\frac{5}{6}} M_2^{\frac{1}{3}} I(1,1).$$

The compactness of the minimizing problem can be proved, but this requires more work...

# Chapter 10

# Blow up: an introduction

Blow up mechanisms are in general poorly understood, but the subject has lead to a tremendous activity in the last twenty years. The scenario leading to the formation of a singularity may be complicated and the phenomenon appears under very different forms: shock waves, turbulence, energy concentration on nonlinear structures, ... The celebrated problem of finite time blow up for the incompressible Navier Stokes which are the basic equations of fluid mechanics is one of the Millenium Clay problem.

In this context, the singularity formation for the (NLS) focusing equation is a fantastic model problem which comprehension has considerably advanced in the last fifteen years, in direct connection with the mathematics developped in these notes.

As an introduction to blow up techniques, we present here the seminal pionnering work of Merle (1992) of classification of the minimal blow up bubble for the mass critical (NLS). This result has been a completely isolated point in the analysis of non linear PDEs until it became in 2006 the corner stone of the revolutionary approach to global existence and scattering for critical non linear dispersive equations known as the Kenig-Merle route map (2006) (see [32], for references).

In order to simplify the exposition as much as possible, we focus in this chapter onto the historical problem of non linear optics:

$$\begin{cases} i\partial_t u + \Delta u + u|u|^2 = 0 \\ u_{|t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^2 \tag{10.1}$$

which is $L^2$ critical in dimension $d = 2$ ie $s_c = 0$.

## 10.1 Critical dynamics and minimal objects

We presented in chapter 7 a first variational characterization of the ground state solitary wave to (6.1) for $s_c < 1$, and hence in particular for (10.1). The stability analysis of chapter 9 however requires the stronger assumption $s_c < 0$, and is false in the critical case $s_c = 0$. In this chapter, we will derive a *dynamical* characterization of the ground state solitary wave as solution to (10.1): it is the *smallest* non linear object, ie the first solution (in term of mass) which does not disperse in large times.

### 10.1.1 Another variational characterization of $Q$

We derive in this section the variational characterization of the ground state solitary wave as an extremizer for a Gagliardo-Niremberg interpolation inequality as discovered by M. Weinstein, [40].

**Proposition 10.1.1** (Best constant in Gagliardo-Nirenberg). *Let*

$$J(u) \stackrel{def}{=} \frac{\|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2}{\|u\|_{L^4}^4}, \quad u \in H^1(\mathbb{R}^2)\backslash\{0\}.$$

*Then*

$$\inf_{u \in H^1\backslash\{0\}} J(u) = J(Q) = \frac{\|Q\|_{L^2}^2}{2}, \tag{10.2}$$

*and the infimum is attained exactly on the family*

$$a_0 Q(\lambda_0 x + x_0) e^{i\gamma_0} \quad \textit{avec} \quad (a_0, \lambda_0, x_0, \gamma_0) \in \mathbb{R}^* \times \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}.$$

*Proof of Proposition 10.1.1.* . This follows a now classical path.

**step 1** Reduction to $\|u\|_{L^2} = \|u\|_{L^4} = 1$. In dimension $d = 2$ and for $p = 4$, the Gagliardo-Nirenberg inequality (4.14) is:

$$\forall u \in H^1, \quad \|u\|_{L^4}^4 \lesssim \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2.$$

Hence

$$J = \inf_{u \in H^1\backslash\{0\}} J(u) > 0. \tag{10.3}$$

An explicit computation reveals

$$J(au(\lambda\cdot)) = J(u), \quad \forall (a, \lambda) \in \mathbb{R}^* \times \mathbb{R}_+^*,$$

and hence adjusting the parameters $a, \lambda$, we have

$$J = \inf_{\|u\|_{L^2}=1,\, \|u\|_{L^4}=1} \|\nabla u\|_{L^2}^2. \tag{10.4}$$

**step 2** Compactness. Let $u \in H^1$, and consider its *distribution function*

$$\mu_u(t) \stackrel{\text{def}}{=} \big|\{|u| > t\}\big|, \qquad t \geq 0.$$

We associate to $u$ its *symmetric rearrangement* $u^*$ which is the unique non increasing spherically symmetric function such that

$$\forall t > 0, \quad \mu_{u^*}(t) = \mu_u(t).$$

The identity (1.16) ensures

$$\forall p \geq 1, \quad \|u^*\|_{L^p} = \|u\|_{L^p}.$$

A non trivial fact which we shall admit is that this transformation makes the kinetic energy decrease [1]

$$\int |\nabla u|^2 \, dx \geq \int |\nabla u^*|^2 \, dx.$$

[1] *Polya-Szego inequality*, [30].

Hence given a minimizing sequence $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_{L^2} = \|u_n\|_{L^4} = 1$, we conclude that $v_n = u_n^*$ is minimizing with spherical symmetry. By lower semi continuity of the kinetic energy (cf (2.4)) and compactness of the *radial* Sobolev embedding $H^1_r \hookrightarrow L^4$, we extract $(v_{\varphi(n)})_{n\in\mathbb{N}}$ with $v_{\varphi(n)} \rightharpoonup v$ in $H^1$ and :

$$J \geq \|\nabla v\|_{L^2}^2, \quad \|v\|_{L^4} = 1 \ \text{ and } \ \|v\|_{L^2} \leq 1.$$

If $\|v\|_{L^2} < 1$ then $J(v) < J$ which contradicts the definition of $J$. Hence the infimum is attained on $v$.

**step 3** Classification of minimizers. If $u$ is a minimizer, so is $v = |u|$ by (7.7). We may therefore first restrict the study to *positive* minimizers. We claim the existence of Lagrange multipliers $\lambda, \mu$ such that

$$\Delta v - \lambda v + \mu v^3 = 0. \tag{10.5}$$

Arguing as in chapter 7, we fix $h \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ and then, for $t \in \mathbb{R}$ close enough to $0$, we let $v_t \stackrel{\text{def}}{=} a_t(v+th)(\lambda_t \cdot)$ with $a_t$ and $\lambda_t$ choisen so that

$$\|v_t\|_{L^2} = 1 \quad \text{et} \quad \|v_t\|_{L^4} = 1.$$

A simple computation using $\|v\|_{L^2} = \|v\|_{L^4} = 1$ reveals:

$$a_t = \frac{\|v+th\|_{L^2}}{\|v+th\|_{L^4}^2} \quad \text{and} \quad \lambda_t = \left(\frac{\|v+th\|_{L^2}}{\|v+th\|_{L^4}}\right)^2.$$

Hence

$$\begin{aligned} \|\nabla v_t\|_{L^2}^2 &= a_t^2\lambda_t^2 \int |(\nabla(v+th))(\lambda_t x)|^2\,dx \\ &= \frac{\|v+th\|_{L^2}^2}{\|v+th\|_{L^4}^4} \int \left(|\nabla v(x)|^2 + 2t\nabla v(x)\cdot\nabla h(x) + t^2|\nabla h(x)|^2\right)dx. \end{aligned}$$

Then using (7.11), (7.12) and (7.13) with $p = 3$,

$$\|\nabla v_t\|_{L^2}^2 = \left(\frac{1+2t\int vh\,dx}{1+4t\int v^3h\,dx}\right)\left(\|\nabla v\|_{L^2}^2 - 2t\int \Delta v\,h\,dx\right) + \mathcal{O}(t^2).$$

Hence

$$\|\nabla v_t\|_{L^2}^2 = \|\nabla v\|_{L^2}^2 + 2\left(\|\nabla v\|_{L^2}^2 \int (v-2v^3)h\,dx - \int \Delta v\,h\,dx\right)t + \mathcal{O}(t^2).$$

This expression ensures that the map $t \mapsto \|\nabla v_t\|_{L^2}^2$ is derivable close to $0$, and since the infimum is attained at $0$, we conclude that (10.5) holds with

$$\lambda = \|\nabla v\|_{L^2}^2 = J \quad \text{and} \quad \mu = 2\lambda.$$

Since $\lambda$ and $\mu$ are striclty positive, we may adjust the constants $a, b > 0$ so that $w \stackrel{\text{def}}{=} av(b\cdot)$ satisfies:

$$\Delta w - w + w^3 = 0, \quad w \geq 0, \quad w \in H^1\backslash\{0\}.$$

We conclude using Theorem 9.2.1 that there exists $x_0 \in \mathbb{R}^2$ such that $w = Q(\cdot - x_0)$.
We now consider $u$ general minimizer, then since $Q$ does not vanish and $u$ is a minimizer, the relation

$$\int |\nabla u|^2 \, dx = \int |\nabla |u||^2 \, dx$$

yields $u = |u|e^{i\gamma}$ for $\gamma \in \mathbb{R}$, and the classification is complete.
It remains to prove (10.2). We mulptiply the $Q$ equation by $Q + x \cdot \nabla Q$ and compute using the Pohozaev identity :

$$\int |\nabla Q|^2 \, dx = \frac{1}{2} \int Q^4 \, dx \quad \text{i.e.} \quad E(Q) = 0. \tag{10.6}$$

Hence $J = J(Q) = \frac{1}{2} \|Q\|_{L^2}^2$. □

We will use the following equivalent formulation of Proposition 10.1.1.

**Corollary 10.1.1** (Lower bound on the energy functional)**.**

$$\forall u \in H^1, \quad E(u) \overset{def}{=} \frac{1}{2} \int \|\nabla u\|_{L^2}^2 - \frac{1}{4} \|u\|_{L^4}^4 \geq \frac{\|\nabla u\|_{L^2}^2}{2} \left[ 1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2} \right] \cdot \tag{10.7}$$

*Moreover,*

$$(E(u) = 0 \quad \textit{and} \quad \|u\|_{L^2} = \|Q\|_{L^2}) \Leftrightarrow u(x) = \lambda_0 Q(\lambda_0 x + x_0) e^{i\gamma_0}$$

*for some* $(\lambda_0, x_0, \gamma_0) \in \mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{R}$.

*Proof of Corollary 10.1.1.* By Proposition 10.1.1:

$$\forall u \in H^1, \quad J(u) = \frac{\|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2}{\|u\|_{L^4}^4} \geq J(Q) = \frac{\|Q\|_{L^2}^2}{2}$$

which implies (10.7). If $E(u) = 0$ and $\|u\|_{L^2} = \|Q\|_{L^2}$, then $J(u) = J(Q)$ and hence $u = a_0 Q(\lambda_0 x + x_0) e^{i\gamma_0}$. The constraint $\|u\|_{L^2} = \|Q\|_{L^2}$ fixes the constant $a_0 = \lambda_0$. □

In other words, the total energy controls the kinetic energy for functions with mass $\|u\|_{L^2} < \|Q\|_{L^2}$, and at the critical level of mass $\|u\|_{L^2} = \|Q\|_{L^2}$, the only (up to symmetries) zero energy function is the ground state solitary wave.

### 10.1.2 Generalized orbital stability

An important corollary of this new variational characterization is the following generalization of Proposition 9.1.2 to the mass critical case.

**Proposition 10.1.2** (Mass critical oribital stability)**.** *Let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $H^1$ *with*

$$\|u_n\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla u_n\|_{L^2} \to \|\nabla Q\|_{L^2} \textit{ and } \limsup_{n\to+\infty} E(u_n) \leq 0. \tag{10.8}$$

*Then up to a subsequence, there exist* $(x_n)_{n\in\mathbb{N}}$ *and* $(\gamma_n)_{n\in\mathbb{N}}$ *elements of respectively* $\mathbb{R}^2$ *and* $\mathbb{R}$ *such that*

$$u_n(\cdot + x_n) e^{i\gamma_n} \to Q \quad \textit{dans} \quad H^1.$$

*Proof of Proposition 10.1.2.* Let us apply the profile decomposition Proposition 9.3.1 to the sequence $u_n$. Up to subsequence, this yields

$$\|Q\|_{L^2}^2 = \|u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + \|v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1) \geq \sum_{j=1}^{\ell} \|V^j\|_{L^2}^2 + o_{n\to+\infty}(1)$$

and hence letting $n \to +\infty$ and then $\ell \to +\infty$:

$$\|Q\|_{L^2}^2 \geq \sum_{j=1}^{+\infty} \|V^j\|_{L^2}^2 \Rightarrow \forall \ell \geq 1, \quad \|V^\ell\|_{L^2} \leq \|Q\|_{L^2}. \tag{10.9}$$

On the other hand,

$$E(u_n) = \sum_{j=1}^{\ell} E(V^j) + \sum_{j=1}^{\ell} \|\nabla v_n^\ell\|_{L^2}^2 + o_{n\to+\infty}(1) + \varepsilon_n^\ell \geq \sum_{j=1}^{\ell} E(V_j)^2 + o_{n\to+\infty}(1) + \varepsilon_n^\ell$$

and hence letting $n \to +\infty$ and then $\ell \to +\infty$ yields:

$$0 \geq \sum_{j=1}^{+\infty} E(V_j).$$

We now invoque (10.9) and the sharp Gagliardo-Nirenberg inequality (10.7) which yield

$$\forall j \geq 1, \quad E(V_j) \geq \|\nabla V^j\|_{L^2}^2 \left(1 - \frac{\|V^j\|_{L^2}^2}{\|Q\|_{L^2}^2}\right) \geq 0$$

and hence necessarily

$$\left| \begin{array}{l} V_j = 0 \ \text{ for } \ j \geq 2 \\ E(V_1) \leq 0, \quad \|V^1\|_{L^2} \leq \|Q\|_{L^2}. \end{array} \right.$$

If $V^1 = 0$, we conclude that up to a subsequence,

$$u_n \to 0 \ \text{ in } \ L^4$$

and then using (10.8)

$$E(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 - \frac{1}{4}\int |u_n|^4\,dx \to \frac{\|\nabla Q\|_{L^2}^2}{2} \quad \text{as} \quad n \to +\infty$$

which contradicts (10.8). Hence $V^1 \neq 0$ from which using (10.1.1):

$$V^1 = \lambda_0 Q(\lambda_0 x + x_0) e^{i\gamma_0}.$$

Moreover

$$\left| \begin{array}{l} u_n(\cdot + x_n^1) \rightharpoonup V^1 \ \text{ in } \ H^1 \\ u_n(\cdot + x_n^1) \to V^1 \ \text{ in } \ L^4 \end{array} \right.$$

and $E(u_n) \to 0$ (because $\forall n,\ E(u_n) \geq 0$) force

$$E(u_n) \to E(V^1) = 0 \Rightarrow \|\nabla u_n\|_{L^2}^2 = \|\nabla Q\|_{L^2}^2 \to \|\nabla V^1\|_{L^2}^2$$

and hence $\lambda_0 = 1$, and

$$u_n(\cdot + x_n^1) \to V^1 = Q(x + x_0)e^{i\gamma_0} \ \text{ in } \ H^1$$

by strong convergence of the norms, and the Proposition is proved. □

### 10.1.3 Minimality of the solitary wave

The variational characterization as a Gagliardo-Nirenberg extremizer implies its dynamical unstability: it is the smallest non scattering element.

**Proposition 10.1.3** ($Q$ is minimal)**.** *Let $u_0 \in H^1$ avec*

$$\|u_0\|_{L^2} < \|Q\|_{L^2}. \tag{10.10}$$

*Then the corresponding solution to* (10.1) *is global in time $\pm\infty$ and disperses as $t \to \pm\infty$: $\exists u_{\pm\infty}$ in $H^1$ such that*

$$\lim_{t\to\pm\infty} \|u - e^{it\Delta}u_{\pm\infty}\|_{H^1} = 0. \tag{10.11}$$

*Proof of Proposition 10.1.3.* Glopal existence follows from the Cauchy theory, and scattering from the pseudo conformal symmetry.

**step 1** Global existence. We observe that (10.7) applied $u(t)$ combined with the conservation of mass and energy implies a uniform bound on the kinetic energy, and the blow up criterion (6.3) yields $u \in \mathcal{C}([0,+\infty), H^1)$.

**step 2** Scattering. We give the proof for $u_0 \in \Sigma$. The proof for $u_0 \in H^1$ only is considerably more complicated. The pseudo conformal invariance (5.23) is *still a symmetry of the non linear problem* in the $L^2$-critical case. Let then

$$u(t,x) = \frac{1}{1+t} v\left(\frac{t}{1+t}, \frac{x}{1+t}\right) e^{i\frac{|x|^2}{4(1+t)}}$$

which solves (10.1) with

$$\forall t \neq -1,\ \|v(T,\cdot)\|_{L^2} = \|u(t,\cdot)\|_{L^2} < \|Q\|_{L^2} \quad \text{with} \ \ T = \frac{t}{1+t}.$$

Hence $v$ is global solution to (10.1) which satisfies

$$v(T,X) \to v(1,X) \ \text{ in } \ H^1 \ \text{ when } \ T \to 1,$$

and (10.11) follows from an explicit computation.

□

## 10.2 Dynamical classification of the solitary wave

We conclude this chapter by a new class of theorem which lie within the class of rigidity theorems[2]. We aim at transforming the variational characterization of the solitary wave (the smallest $H^1$ solution in $L^2$ with zero energy) into a dynamical classification of the solitary wave: it is up to symmetries the unique solution to (NLS) with mass $\|Q\|_{L^2}$ which does not disperse. This should be thought of in the following way: being the *first and minimal* non trivial solution of the non linear flow is a very rigid property and forces a very particular non linear stucture to emerge, here the solitary wave. Should the problem *not admit* such very special solutions, then there can be no such minimal first non scattering solution, and hence all solutions must disperse and scatter: this is the Kenig Merle route map [21]. The heart of the proof is the classification of minimal elements.

[2]See [32] for an elementary introduction.

### 10.2.1 The minimal blow up bubble

The global existence criterion (10.10) is optimal in the following two ways: first the solitary wave $u(t,x) = Q(x)e^{it}$ is global and non dispersive with minimal mass $\|u(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$; second there exists a blow up solution with minimal mass $\|Q\|_{L^2}$. Indeed, the conformal invariance (5.23) being still a symmetry of the non linear flow in the mass critical case, and exchanging the roles of and $t$ and $T = \frac{t}{1+t}$, we obtain

$$v(T,X) = \frac{1}{1-T} u\left(\frac{T}{1-T}, \frac{X}{1-T}\right) e^{-i\frac{|X|^2}{4(1-T)}}.$$

Applied to the solitary wave $u(t,x) = Q(x)e^{it}$ produces the explicit solution

$$S(t,x) \equiv \frac{1}{1-t} Q\left(\frac{x}{1-t}\right) e^{-i\frac{|x|^2}{4(1-t)} + \frac{it}{1-t}}, \tag{10.12}$$

which emerges from the data at time $0$:

$$S(0,x) = Q(x)e^{-i\frac{|x|^2}{4}}.$$

The solution $S$ blows up at time $t = 1$ at the speed

$$\|\nabla S(t,\cdot)\|_{L^2} = \frac{c}{1-t},$$

but is globally defined and scatters as $t \to -\infty$. Moreover, the conformal invariance being an $L^2$ isometry, this solution has miminal mass

$$\|S(t,\cdot)\|_{L^2} = \|Q\|_{L^2}.$$

### 10.2.2 Uniqueness of the minimal blow up bubble

A spectacular property of the $S$ solution is that at the time of the singularity, all the mass of the solution concentrates at the origin as (10.12) easily implies

$$|S(t,\cdot)|^2 \rightharpoonup \|Q\|_{L^2}^2 \delta_{x=0} \ \text{ in } \ \mathcal{D}'(\mathbb{R}^2). \tag{10.13}$$

Hence *all the mass* available at $t = 0$ has focused into the singularity. Like the solitary wave $Q(x)e^{it}$ which does not loose energy as it propagates, the minimal blow up bubble is *non dispersive* and *compact in* $H^1$ up to symmetries, ie it does not eject any energy during the evolution. This behaviour must be *non generic* and such objects are rigid: these are the bubbles of energy which drive the non linear flows, anything else should be linear radiation, this is the *soliton resolution conjecture.* Hence the *dynamical* classification of these *compact* bubbles is a fundamental step towards the understanding of all solutions, and a pioneering result in this direction is:

**Theoreme 10.2.1** (Classification of the minimal bubble, F. Merle (1992) [29])**.** *Let $u_0 \in H^1$ with*

$$\|u_0\|_{L^2} = \|Q\|_{L^2}.$$

*If the corresponding solution $u \in \mathcal{C}([0,T[; H^1)$ blows up in finite time $T < +\infty$, then*

$$u(t,x) \equiv S(t,x)$$

*up to the symmetries of the flow*[3].

[3] that is scaling, phase and translations, see Proposition 6.2.1.

Through the conformal invariance, this yields equivalently the dynamical classification of the solitary wave.

**Corollary 10.2.1** (Dynamical classification of the solitary wave)**.** *Let* $u_0 \in H^1$ *with*

$$\|u_0\|_{L^2} = \|Q\|_{L^2}.$$

*If* $u$ *is neither the solitary wave nor* $S$ *up to symmetries, then* $u$ *is global and scattering in both directions of time.*

In other words, Proposition 10.1.3 extends to the limiting case $\|u_0\|_{L^2} = \|Q\|_{L^2}$, modulo the consideration of two exceptional solutions: the minimal dynamics of the solitary wave and $S$ which are $H^1$ compact. The proof is the starting point of the analysis of *all* data $u_0$ with $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$, and $0 < \alpha^* \ll 1$, for which most dynamics have now been classified.

### 10.2.3 Proof of Theorem10.2.1

Let $u_0 \in H^1$ be a minimal mass blow up solution: $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and $u \in \mathcal{C}([0,T[;H^1)$ the corresponding solution to (10.1). We assume finite time blow up $T < +\infty$.

**step 1** Orbital stability and $H^1$ compactness. Let us renormalize the solution

$$\lambda(t) \stackrel{\text{def}}{=} \frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t)\|_{L^2}}. \tag{10.14}$$

Then the blow up criterion (6.3) ensures

$$\lim_{t \nearrow T} \lambda(t) = 0. \tag{10.15}$$

Let

$$v(t,x) \stackrel{\text{def}}{=} \lambda(t) u(t, \lambda(t)x),$$

then by (10.14):

$$\|\nabla v(t,\cdot)\|_{L^2} = \|\nabla Q\|_{L^2}.$$

By conservation of mass

$$\|v(t,\cdot)\|_{L^2} = \|u(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$$

and energy with (10.15) :

$$E(v(t)) = \lambda^2(t) E(u(t)) = \lambda^2(t) E(u_0) \to 0 \ \text{ as } \ t \to T.$$

We conclude using Proposition 10.1.2 that there exists $(x(t), \gamma(t)) \in \mathbb{R}^2 \times \mathbb{R}$ such that:

$$v(t, \cdot + x(t)) e^{i\gamma(t)} \to Q \ \text{ dans } \ H^1 \ \text{as} \ t \to T.$$

Coming back to $u$, we conclude

$$u(t,x) = \frac{1}{\lambda(t)} [Q + \varepsilon] \left( \frac{x - x(t)}{\lambda(t)} \right) e^{i\gamma(t)} \ \text{ with } \ \lim_{t \to T} \|\varepsilon(t,\cdot)\|_{H^1} = 0. \tag{10.16}$$

In other words, up to renormalization, $u$ has a strong $H^1$ limit as $t \to T$, and does not eject mass: we say that the flow is *non dispersive* or *compact in* $H^1$.

**step 2** A Cauchy-Schwarz inequality. From (10.7), let $u \in H^1$ with $\|u\|_{L^2} < \|Q\|_{L^2}$, then the total energy controls the kinetic energy. We claim a slightly weaker statement at the minimal level: let $w \in H^1$ with $\|w\|_{L^2} \leq \|Q\|_{L^2}$ and $\psi$ a smooth function, then

$$\left|\mathrm{Im}\left(\int \overline{w}\,\nabla w \cdot \nabla\overline{\psi}\,dx\right)\right|^2 \leq 2E(w)\int |\nabla\psi|^2|w|^2\,dx. \tag{10.17}$$

Indeed, we compute for all $a \in \mathbb{R}$ :

$$E(we^{ia\psi}) = E(w) + a\,\mathrm{Im}\left(\int \overline{w}\,\nabla w \cdot \nabla\overline{\psi}\,dx\right) + \frac{1}{2}a^2\int |w|^2|\nabla\psi|^2\,dx.$$

Now $\|we^{ia\psi}\|_{L^2} = \|Q\|_{L^2}$, implies $E(we^{ia\psi}) \geq 0$ by (10.7), which implies (10.17) by rewriting down the discriminant of the order two polynomial in $a$.

**step 3** $L^2$ tightness and control of the concentration point. We now inject the first dynamical information: we localize the mass conservation and claim that it implies the $L^2$ tightness of the flow

$$\forall \varepsilon > 0,\ \exists R > 0\ t.q.\ \forall t \in [0,T[,\quad \int_{|x|\geq R} |u(t,x)|^2 dx < \varepsilon. \tag{10.18}$$

Observe by (10.16) that this immediately implies the control of the concentration point

$$\limsup_{t\to T} |x(t)| < +\infty. \tag{10.19}$$

*Proof of* (10.18). Let $\chi$ smooth with spherical symmetry with $\chi(r) = 0$ for $r \leq \frac{1}{2}$ and $\chi(r) = 1$ for $r \geq 1$. Let $R > 0$ and $\chi_R(x) \overset{\text{def}}{=} \chi\left(\frac{x}{R}\right)$. We compute the evolution of the localized mass:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int \chi_R|u|^2\,dx &= \mathrm{Re}\left(\int \chi_R\partial_t u\overline{u}\,dx\right) = \mathrm{Im}\left(\int i\partial_t u\chi_R\overline{u}\,dx\right)\\
&= -\mathrm{Im}\left(\int [\Delta u + u|u|^2)\chi_R\overline{u}\,dx\right) = \mathrm{Im}\left(\int \nabla u \cdot \nabla(\chi_R\overline{u})\,dx\right)\\
&= \mathrm{Im}\left(\int \nabla u \cdot \nabla\chi_R\overline{u}\,dx\right).
\end{aligned} \tag{10.20}$$

We conclude using (10.17) and the conservation of mass:

$$\left|\frac{d}{dt}\int \chi_R|u|^2\,dx\right| \lesssim \sqrt{E(u_0)}\left(\int |\nabla\chi_R|^2|u|^2\,dx\right)^{\frac{1}{2}} \lesssim \frac{\sqrt{E(u_0)}\|u_0\|_{L^2}}{\sqrt{R}}.$$

We integrate bewteen $0$ and $t < T$:

$$\forall t \in [0,T[,\quad \int \chi_R|u(t,x)|^2 dx \leq \int \chi_R|u_0(x)|^2\,dx + \frac{C(u_0)T}{\sqrt{R}},$$

and (10.18) is proved. Note that this step uses the finite time blow up assumption.

**step 4** Improved regularity. We now enter the heart of the proof: the improved regularity. A solution to a non linear dispersive PDE's inherits the regularity of its Cauchy data. But the key to the classification of minimal bubbles is to integrate the flow *from the singularity* and use the minimality to gain regularity. Here more precisely we will show that necessarily the

solution gains integrability at infinity $u \in \mathcal{C}([0,T[;\Sigma)$: since there is no mass at blow up time away from the origin, there could not be "too much" mass at infinity at initial time.
By (10.19), up to possibly translating by a fixed vector, there exists $t_n \uparrow T$ with :

$$x(t_n) \to 0 \quad \text{in} \quad \mathbb{R}^2. \tag{10.21}$$

Let $\psi$ be a radially symmetric function $\psi(r) = r^2$ for $r \leq 1$, $\psi(r) = 8$ for $r \geq 2$ and $|\nabla \psi|^2 \leq C\psi$. Let $A > 0$ and $\psi_A(r) \stackrel{\text{def}}{=} A^2 \psi(\frac{r}{A})$. Then there exists a constant $C$ independant of $A$ such that

$$|\nabla \psi_A|^2 \leq C\psi_A. \tag{10.22}$$

We estimate using (10.20), (10.17) and (10.22):

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int \psi_A |u|^2\,dx\right| &= \left|\mathrm{Im}\int (\nabla \psi_A \cdot \nabla u\,\overline{u})\,dx\right| \lesssim \sqrt{E_0}\left(\int |\nabla \psi_A|^2 |u|^2\,dx\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{E_0}\left(\int \psi_A |u|^2\,dx\right)^{\frac{1}{2}}\end{aligned}$$

and hence

$$\left|\frac{d}{dt}\sqrt{\int \psi_A |u|^2}\right| \lesssim \sqrt{E_0}. \tag{10.23}$$

Now from (10.21) and (10.16):

$$\int \psi_A |u(t_n)|^2\,dx \to 0 \quad \text{as} \quad n \to +\infty.$$

Hence, integrating (10.23) on $[t, t_n]$ and letting $n$ vers $+\infty$, we obtain

$$\forall t \in [0,T[,\quad \sqrt{\int \psi_A |u(t)|^2\,dx} \leq C(E_0)(T-t),$$

where we used finite time blow up again. Since the right hand side is independent of $A$, Fatou's lemma ensures letting $A \to +\infty$:

$$\forall t \in [0,T[,\quad u(t) \in \Sigma \quad \text{with} \quad \int |x|^2 |u(t,x)|^2 dx \leq C(E_0)(T-t). \tag{10.24}$$

**step 5** Conformal invariance and conclusion. The last step is algebra. The bound (10.24) implies

$$\int |x|^2 |u(t,x)|^2 dx \to 0 \quad \text{as} \quad t \to T.$$

Let now

$$v(t,x) = \left(\frac{T}{T+t}\right) u\left(\frac{tT}{T+t}, \frac{Tx}{T+t}\right) e^{i\frac{|x|^2}{4(T+t)}}.$$

Then

$$\|v(t,\cdot)\|_{L^2} = \|u(t,\cdot)\|_{L^2} = \|Q\|_{L^2}$$

and a direct computation ensures:

$$E(v) = \frac{1}{8}\lim_{t \nearrow T}\int |x|^2 |u(t,x)|^2\,dx = 0.$$

We conclude using Corollary 10.1.1 that $v \equiv Q$ up to symmetries, and hence $u \equiv S$ up to symmetries. ■

## 10.3 Exercices

**Exercice 10.1.** Generalize Proposition 10.1.1 with $(p,d)$ such that $s_c<1$ and obtain the solitary wave as the extremizer of a suitable Gagliardo-Nirenberg inequality.

**Exercice 10.2.** Let $u_0\in H^1$ with $\|u_0\|_{L^2}=\|Q\|_{L^2}+\alpha^*$ for some $0<\alpha^*\ll 1$. We assume that the corresponding solution $u\in\mathcal{C}([0,T[;H^1)$ of the $L^2$-critical $(NLS)$ (i.e. $p=1+\frac{4}{d}$) blows up in finite time. We define $v$ through the renormalization :

$$u(t,x)=\frac{1}{\lambda(t)^{\frac{d}{2}}}v\left(\frac{x}{\lambda(t)}\right)\ \text{ with } \lambda(t)\stackrel{\text{def}}{=}\frac{\|\nabla Q\|_{L^2}}{\|\nabla u(t)\|_{L^2}}.$$

(*i*) Show that

$$\lim_{t\nearrow T}E(v(t))=0.$$

(*ii*) Show that there exist $(x(t),\gamma(t))\in\mathbb{R}^d\times\mathbb{R}$ such that

$$\forall x\in\mathbb{R}^d,\ v(t,x+x(t))e^{i\gamma(t)}=Q(x)+\varepsilon(t,x)$$

with

$$\sup_{t\in[0,T[}\|\varepsilon(t)\|_{H^1}=o(1)\ \text{ as }\ \alpha^*\to 0.$$

Hint: argue by contradiction and use Proposition 10.1.2.

**Exercice 10.3** (Mass concentration for the cubic focusing $(NLS)$ in dimension $2$)**.** Consider (10.1) with data $u_0\in H^1_r$. We suppose that the corresponding radial solution $u$ blows up in finite time $0<T<+\infty$. We will show that *the critical norm must concentrate*: :

$$\forall R>0,\quad \liminf_{t\nearrow T}\int_{|x|\le R}|u(t_n,x)|^2dx\ge\|Q\|^2_{L^2}.$$

We argue by contradiction and assume that there exists $\varepsilon,R>0$ and a sequence $t_n\to T$ with

$$\limsup_{n\to+\infty}\int_{|x|\le R}|u(t_n,x)|^2dx<\|Q\|^2_{L^2}-\varepsilon.$$

(*i*) Let

$$\lambda(t)\stackrel{\text{def}}{=}\frac{1}{\|\nabla u(t)\|_{L^2}}.$$

Show that

$$\lim_{t\uparrow T}\lambda(t)=0.$$

(*ii*) Let $v_n(x)=\lambda_n u(t_n,\lambda_n x)$ with $\lambda_n=\lambda(t_n)$. Show that $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1$.

(*iii*) Compute $E(v_n)$.

(*iv*) Let $v$ be a weak limit extracted from $(v_n)_{n\in\mathbb{N}}$. Show that $v$ is non zero.

(*v*) Show that

$$E(v)\le 0\ \text{ and }\ 0<\int|v|^2\,dx\le\int Q^2\,dx\ -\varepsilon,$$

and conclude.

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