Introduction to non linear Analysis

Example sheet n^o 3 - NLS

Exercices to be done : 1-2-3.

Exercice 1 (A symmetry of the harmonic oscillator). *We consider the cubic non linear harmonic oscillator in dimension 2*

$$i\partial_t u + \Delta u - |x|^2 u + u|u|^2 = 0.$$

1. *Define the renormalization*

$$\left|\begin{array}{l} u(t,x) = \frac{1}{L} w(s,y) \\ w(s,y) = e^{-i\frac{b(s)|y|^2}{4}} v(s,y) \\ y = \frac{x}{L}, \quad \frac{ds}{dt} = \frac{1}{L^2}. \end{array}\right.$$

   Show that $v(s,y)$ satisfies the same equation iff

$$\left|\begin{array}{l} \frac{L_s}{L} + b = 0 \\ \frac{b_s}{4} - \left(\frac{b^2}{4} + L^4\right) - \frac{b}{2}\frac{L_s}{L} = -1 \end{array}\right. \tag{0.1}$$

   where $f_s = \frac{df}{ds}$.

2. *Integrate the dynamical system* (0.1) *in time t. (hint : look for a conserved quantity and draw the phase portrait).*

Exercice 2 (Local existence in H^2 for cubic (NLS) in dimension 2). *We consider*

$$(S_\varepsilon) \qquad i\partial_t u + \Delta u + \varepsilon|u|^2 u = 0 \quad \textit{dans} \ \ \mathbb{R}\times\mathbb{R}^2, \qquad \varepsilon \in \{-1, 1\}$$

with data $u_0 \in H^2(\mathbb{R}^2)$.

Let $E_T = \mathcal{C}([0,T]; H^2)$ and $u_L = S(t)u_0$. For $u \in E_T$, we let

$$\forall t \in [0,T], \ \Phi(u)(t) = u_L(t) + i\varepsilon \int_0^t S(t-\tau)((|u|^2 u(\tau))\, d\tau.$$

1. *Show that $H^2(\mathbb{R}^2)$ is stable by product.*
2. *Show that Φ is well defined from E_T into E_T, and there exist two constants $C_1, C_2 > 0$ such that for all u, v in $B_{E_T}(u_L, R)$,*

$$\|\Phi(u) - u_L\|_{E_T} \le C_1 T(R^3 + \|u_0\|_{H^2}^3) \ \textit{and} \ \|\Phi(u) - \Phi(v)\|_{E_T} \le C_2 T(R^2 + \|u_0\|_{H^2}^2)\|u - v\|_{E_T}.$$

3. *Conclude that there exists $c > 0$ and a time $T \ge c/\|u_0\|_{H^2}^2$ such that Φ has a fixed point in E_T.*
4. *Conclude that there exists $T^* > 0$ such that (S_ε) with data $u_0 \in H^2$ has a unique maximal solution $u \in \mathcal{C}([0,T^*[; H^2) \cap \mathcal{C}^1([0,T^*[; L^2)$.*
5. *In this question, we look for a blow up criterion.*
   (a) *Prove the Gagliardo-Nirenberg estimate :*

$$\forall u \in H^2(\mathbb{R}^2), \ \|\partial_1 u\|_{L^4}^2 \le 3\|u\|_{L^\infty}\|\partial_{11}^2 u\|_{L^2}.$$

   (b) *Prove the* tame estimate *:*

$$\forall (u,v) \in H^2 \times H^2, \ \|uv\|_{H^2} \le C_0\big(\|u\|_{L^\infty}\|v\|_{H^2} + \|v\|_{L^\infty}\|u\|_{H^2}\big)$$

   for some universal constant C_0.

(c) *Show that there exists* $C > 0$ *universal such that for all solution* $u \in E_T$ *of* (S_ε) :

$$\forall t \in [0,T], \ \|u(t)\|_{H^2} \leq \|u_0\|_{H^2} + C\int_0^t \|u\|_{L^\infty}^2 \|u\|_{H^2}\, d\tau.$$

(d) *Conclude that* $T^* < +\infty$ *implies* $\int_0^{T^*} \|u(t)\|_{L^\infty}^2\, dt = +\infty$. *For* $\varepsilon = -1$, *does this allow to conclude* $T^* = +\infty$?

6. *Using Theorem 6.2.1 in the notes (gwp in* H^1*), prove global existence in* H^2 *for* $\varepsilon = -1$.

Exercice 3 (An upper bound on blow up rate for (NLS)). *We work in* $\mathbb{R}^2$. *Let the focusing (NLS)*

$$\left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0,x) = u_0(x) \end{array} \right. , \quad x \in \mathbb{R}^2, \quad 3 < p < 5.$$

Let H^1_r *be the set of* H^1 *functions with radial symmetry, then the Cauchy problem is well posed in* H^1_r. *We pick* $u_0 \in H^1_r$ *and assume that the solution blows up in finite time* $0 < T < +\infty$. *The aim of this problem is to derive an upper bound on* $\|\nabla u(t)\|_{L^2}$ *as* $t \uparrow T$.
Integration by parts should be done without boundary terms (without justification). We let

$$s_c = 1 - \frac{2}{p-1}$$

and E_0 *be the energy of the data. We recall Young's inequality :*

$$|xy| \leq \frac{1}{p}\left(\frac{|x|}{A}\right)^p + \frac{(A|y|)^{p'}}{p'}, \quad 1 \leq p, p' \leq +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad A > 0. \tag{0.2}$$

1. *Let* $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ *with spherical symmetry, prove the formulas :*

$$\frac{1}{2}\frac{d}{d\tau}\int \chi|u|^2 = Im\left(\int \nabla\chi \cdot \nabla u \overline{u}\right),$$

and

$$\frac{1}{2}\frac{d}{d\tau} Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right) = \int \chi''|\nabla u|^2 - \frac{1}{4}\int \Delta^2\chi|u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\chi|u|^{p+1}.$$

2. *Prove that for all* $u \in H^1_r$,

$$\forall R > 0, \quad \|u\|_{L^\infty(r\geq R)}^2 \leq \frac{2}{R}\|u\|_{L^2}\|\nabla u\|_{L^2}.$$

3. *Let* $R > 0$, $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ *with spherical symmetry and*

$$\psi(x) = \left| \begin{array}{ll} \frac{|x|^2}{2} & pour \ \ |x| \leq 2 \\ 0 & pour \ \ |x| \geq 3 \end{array} \right. .$$

Let

$$\chi(x) = \psi_R(x) = R^2\psi\left(\frac{x}{R}\right),$$

show that

$$c(d,p)\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(d,p)\left[|E_0| + \int_{|x|\geq R}|u|^{p+1} + \frac{1}{R^2}\int_{2R\leq|x|\leq 3R}|u|^2\right]$$

for some constants $c(d,p), C(d,p) > 0$ *independant of* R.

4. *Prove using (0.2) that :*

$$\frac{c(d,p)}{2}\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(u_0,d,p)\left[1 + \frac{1}{R^2} + \frac{1}{R^{\frac{2}{\alpha}}}\right] \tag{0.3}$$

with

$$\alpha = \frac{5-p}{p-1}.$$

5. *Integrate in time (0.3) and prove :* $\forall 0 < t_0 < t_2 < T$,

$$\int_{t_0}^{t_2} (t_2 - t)\|\nabla u(t)\|_{L^2}^2 dt \le C(u_0, d, p)\left[\frac{(t_2 - t_0)^2}{R^{\frac{2}{\alpha}}} + R(t_2 - t_0)\|\nabla u(t_0)\|_{L^2} + R^2\right].$$

6. *Choose* $R = (T - t_0)^{\frac{\alpha}{1+\alpha}}$ *and conclude that for* t *close enough to* T *:*

$$\int_{t_0}^{T} (T - t)\|\nabla u(t)\|_{L^2}^2 dt \le C(d, p, u_0)(T - t)^{\frac{\alpha}{2+\alpha}} + (T - t_0)^2\|\nabla u(t_0)\|_{L^2}^2.$$

7. *Show that for* t *close enough to* T *:*

$$\int_{t_0}^{T} (T - t)\|\nabla u(t)\|_{L^2}^2 dt \le C(d, p, u_0)(T - t)^{\frac{2\alpha}{1+\alpha}}.$$

8. *Conclude that there exists a sequence* $t_n \to T$ *such that*

$$\|\nabla u(t_n)\|_{L^2} \le \frac{C(d, p, u_0)}{(T - t_n)^{\frac{1}{1+\alpha}}}.$$

   (remark : this bound is sharp !).

9. *Open problem : prove any bound on blow up rate in the critical case* $p = 3$ *!*