Introduction to non linear Analysis

Example sheet n° 2 - Weak convergence and linear dispersion

Exercices to be done : 1-2-3.

Exercice 1 (A precised Gagliardo-Nirenberg inequality). *Let $d \geq 1$. Let $2 < p < 2^* - 1$ with*

$$2^* = \left| \begin{array}{ll} +\infty & \text{for } d = 1,2 \\ \frac{d+2}{d-2} & \text{for } d \geq 3. \end{array} \right.$$

1. *Let a sequence $u_n \in H^1(\mathbb{R}^d)$ with $u_n \to 0$ in $H^1(\mathbb{R}^d)$, show that $u_n \to 0$ in $L^p(\mathbb{R}^d)$.*
2. *Let $\phi \in H^1(\mathbb{R}^d)$ and $x_n \in \mathbb{R}^d$ with $\lim_{n\to+\infty} |x_n| = +\infty$, let $u_n(x) = \phi(x - x_n)$. Show that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^d)$ as $n \to +\infty$. Do we have $u_n \to 0$ in $L^p(\mathbb{R}^d)$?*
3. *Let $\mathbf{u} = (u_n)_{n\geq 1}$ be a bounded sequence in H^1 and $\mathcal{V}(\mathbf{u})$ be the subset of all possible weak H^1 limit of the transaltes of u_n : $V \in \mathcal{V}(\mathbf{u})$ iff there exists a subsequence $\phi(n)$ and $x_n \in \mathbb{R}^d$ such that*

$$u_{\phi(n)}(\cdot - x_n) \rightharpoonup V \quad \text{in} \quad H^1(\mathbb{R}^d).$$

   Show that $\mathcal{V}(\mathbf{u})$ is a bounded subset of $H^1(\mathbb{R}^d)$. We therefore define

$$\eta(\mathbf{u}) = \sup_{V\in\mathcal{V}(\mathbf{u})} \|V\|_{H^1}.$$

4. *Let $f \in H^1(\mathbb{R}^d)$ and $u_n = \frac{1}{n^{\frac{d}{2}}} f\left(\frac{x}{n}\right)$. Compute $\eta(\mathbf{u})$. Show that u_n has a limit in L^p for $2 < p < 2^* - 1$. Does u_n converge to 0 in L^2 ?*
5. *We now assume that*

$$\eta(\mathbf{u}) = 0.$$

   Our aim is to show the compactness statement

$$u_n \to 0 \quad \text{in} \quad L^p \quad \text{for} \quad 2 < p < 2^* - 1.$$

   Let us fix $\chi \in \mathcal{S}(\mathbb{R}^d)$ with

$$\widehat{\chi}(\xi) = \left| \begin{array}{ll} 1 & \text{for } |\xi| \leq 1 \\ 0 & \text{for } |\xi| \geq 2 \end{array} \right. , \quad |\widehat{\chi}| \leq 1,$$

   and given $R > 0$, let

$$\chi_R(x) = R^d \chi(Rx).$$

   Let the low-high frequency splitting

$$u_n = u_n^{(1)} + u_n^{(2)}, \quad \left| \begin{array}{l} \widehat{u_n^{(1)}} = \widehat{v_n^\ell}\widehat{\chi_R} \\ \widehat{u_n^{(2)}} = \widehat{v_n^\ell}(1 - \widehat{\chi_R}) \end{array} \right.$$

   Let s be given by $-s + \frac{d}{2} = \frac{d}{p}$. Show that $0 < s < 1$.
6. *Show that there exists $C_{d,p,\chi} > 0$ such that*

$$\forall n \geq 1, \quad \forall R > 0, \quad \|u_n^{(2)}\|_{L^p} \leq \frac{C_{d,p,\chi}}{R^{1-s}}.$$

7. *Show that there exists $C_{d,p,\chi} > 0$ such that*

$$\forall n \geq 1, \qquad \forall R > 0, \quad \|u_n^{(1)}\|_{L^p} \leq C_{d,p,\chi} \|\chi_R \star u_n\|_{L^\infty}^{1-\frac{2}{p}}.$$

8. *Show that*

$$\forall R > 0, \quad \lim_{n \to +\infty} \|\chi_R \star u_n\|_{L^\infty} = 0.$$

9. *Show that* $u_n \to 0$ *in* $L^p(\mathbb{R}^d)$.

Exercice 2 (Cauchy problem for (NLS) in $\mathbb{R}$)**.** *Let* $S(t)$ *be the linear Schrödinger semi group on* $\mathbb{R}$*. Show that there exists* $\alpha_p > 0$ *and* $C_p > 0$*, such that the following holds : for all* $u_0 \in H^1(\mathbb{R})$*, let* $T = \frac{C_p}{\|u_0\|_{H^1}^\alpha}$*, then the map*

$$\Phi(u)(t,x) = S(t)u_0 + \int_0^t S(t-s)(u|u|^2(s,\cdot))ds$$

is a contraction mapping in the Banach space $E = L^\infty_{[0,T]}H^1_x$ *equipped with the norm* $\|u\|_E = \sup_{t\in[0,T]} \|u(t,\cdot)\|_{H^1_x}$.

Exercice 3 (Dispersion for the free transport)**.** *Let the transport equation describing the evolution of the microscopic density* $f(t,x,v) \in \mathbb{R}^+$ *of free particules which are at* $x \in \mathbb{R}^d$ *with the speed* $v \in \mathbb{R}^d$ *at time* $t \in \mathbb{R}$ *:*

$$(T) \qquad \begin{cases} \partial_t f + v \cdot \nabla_x f = 0, \\ f_{|t=0} = f_0. \end{cases}$$

1. *Assume* $f_0 = f_0(x,v)$ *is differentiable, compute the solution to* (T).
2. *If* f_0 *is moreover integrable, show that the total density is converved*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,dx\,dv = \int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\,dx\,dv.$$

3. *We define the macroscopic density* $\rho(t,x) \stackrel{def}{=} \int_{\mathbb{R}^d} f(t,x,v)\,dv$*. Show the pointwise decay :*

$$\|\rho(t,\cdot)\|_{L^\infty} \le \frac{1}{|t|^d}\|\sup_v f_0(\cdot,v)\|_{L^1} \quad \text{for all } t \neq 0.$$

Exercice 4 (Wave equation)**.** *Let the free wave equation*

$$(W) \qquad \begin{cases} \Box u = 0 \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) \end{cases}$$

where $\Box \stackrel{def}{=} \partial_t^2 - \Delta$ *and where* $u = u(t,x) \in \mathbb{R}$, $(t,x) \in \mathbb{R}\times\mathbb{R}^d$.

1. *For* $d = 1$ *and* $(u_0,u_1) \in C^2 \times C^1$*, show that the* C^2 *solution is given by* d'Alembert's formula *:*

$$u(t,x) = \frac{1}{2}\left(u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(y)\,dy\right).$$

Do we have pointwise decay in time ?

2. *For* $d = 3$*, we recall that the solution is given by*

$$u(t,x) = \frac{1}{4\pi}\left(\frac{1}{t}\int_{S(x,t)} u_1(\sigma)\,d\sigma + \frac{d}{dt}\left(\frac{1}{t}\int_{S(x,t)} u_0(\sigma)\,d\sigma\right)\right)$$

wher $S(x,t)$ *is the sphere of center* x *and radius* t*. Assume for simplicity* $u_0 \equiv 0$*, then show :*

$$\|u(t)\|_{L^\infty} \le C\frac{\|\nabla u_1\|_{L^1}}{|t|} + \frac{\|u_1\|_{L^1}}{t^2}.$$

Exercice 5 (Oscillatory integrals)**.** *Let* $a \in \mathcal{D}(\mathbb{R})$ *and* Φ *a* C^2 *function such that for some* $c_0 > 0$ *:*

$$\forall x \in \text{Supp } a, \ \Phi''(x) \ge c_0.$$

For $t \in \mathbb{R}$, *we define the oscillatory integral*

$$I(t) \stackrel{def}{=} \int_{\mathbb{R}} e^{it\Phi(x)} a(x)\, dx.$$

For $t \neq 0$, *we define the differential operator* $\mathcal{L}_t$ *acting on derivable functions* b *by*

$$\mathcal{L}_t b(x) \stackrel{def}{=} \frac{1}{1 + t\big(\Phi'(x)\big)^2}\big(b(x) - i\Phi'(x) b'(x)\big).$$

1. *Using* $\mathcal{L}_t$, *show that* $I(t) = I_1(t) + I_2(t)$ *with*

$$I_1(t) \stackrel{def}{=} \int e^{it\Phi(x)} \frac{i\Phi'(x)}{1 + t(\Phi'(x))^2} a'(x)\, dx \quad \text{and}$$

$$I_2(t) \stackrel{def}{=} \int \frac{e^{it\Phi(x)}}{1 + t(\Phi'(x))^2}\Big(1 + i\Phi''(x) - 2i\frac{t(\Phi'(x))^2\Phi''(x)}{1 + t(\Phi'(x))^2}\Big) a(x)\, dx.$$

2. *Noticing that for* $x \in \operatorname{Supp}\ a$,

$$\frac{1}{1 + t(\Phi'(x))^2} \leq \frac{1}{c_0} \frac{\Phi''(x)}{1 + t(\Phi'(x))^2},$$

show that

$$|I_2(t)| \leq \frac{\pi}{2}\left(\frac{1}{c_0} + 3\right) \frac{1}{|t|^{\frac{1}{2}}} \|a'\|_{L^1(\mathbb{R})}.$$

3. *Conclude that there exists* $C_0(c_0)$ *such that*

$$|I(t)| \leq \frac{C_0}{|t|^{\frac{1}{2}}} \|a'\|_{L^1}.$$

4. *Application : Consider the* Airy equation

$$\partial_t u + \partial^3_{xxx} u = 0$$

with data u_0 *integrable and with Fourier transform supported in*

$$[-2, -1/2] \cup [1/2, 2].$$

   (a) *Show that the* L^2 *norm is conserved. Write* $u(t) = k_t \star u_0$ *for a suitable function* k_t *and conclude*

$$\|u(t)\|_{L^\infty} \leq C|t|^{-\frac{1}{2}} \|u_0\|_{L^1}.$$

   (b) *What kind of* L^p–$L^{p'}$ *estimate do we obtain if* $\widehat{u}_0$ *is supported in the set* $[-2\lambda, -\lambda/2] \cup [\lambda/2, 2\lambda]$ *?*
   Hint : use the fact that if φ *is smooth with support in* $\{\frac{1}{3} \leq |\xi| \leq 3\}$ *and equal to* 1 *on* $\{\frac{1}{2} \leq |\xi| \leq 2\}$, *then* $\widehat{u}_0 = \varphi \widehat{u}_0$.