Introduction to non linear Analysis- Weak convergence and Sobolev spaces

Exercices to be done 2-3-4-6-7.

Exercice 1 (Fundamental solution of Helmoltz)**.** *Let $d \geq 1$.*

1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ with spherical symmetry, show that its Fourier transform also has spherical symmetry*
2. *Let $\lambda > 0$ and $E : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto \frac{e^{-\sqrt{\lambda}\|x\|}}{\|x\|}$, show that $E \in \mathcal{S}'(\mathbb{R}^3)$ and compute its Fourier transform.*
3. *Given $f \in \mathcal{S}(\mathbb{R}^3)$, $\lambda > 0$, solve the Helmoltz equation in $\mathcal{S}'(\mathbb{R}^3)$*
$$(-\Delta + \lambda)u = f$$
and give the representation formula both in Fourier and space variables. Show that for all $s \in \mathbb{R}$, the map $f \mapsto u$ is continuous from H^s into H^{s+2}.

Exercice 2 (Uniform approximation of L^p functions)**.** *Let $d \geq 1$. Let $1 \leq p < +\infty$.*

1. *Show that*
$$\forall h \in \mathbb{R}^d, \quad \forall x \in \mathbb{R}^d, \quad \forall f \in \mathcal{D}(\mathbb{R}^d), \quad |f(x+h) - f(x)| \leq c|h|^p \int_0^1 |\nabla f(x+th)|^p dt.$$
2. *Show that*
$$\forall h \in \mathbb{R}^d, \quad \forall f \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |f(x+h) - f(x)|^p \leq C|h|^p \|\nabla f\|^p_{L^p(\mathbb{R}^d)}.$$
3. *Let $\zeta \in C_c^\infty(\mathbb{R}^d)$ with*
$$Supp(\zeta) \subset \{|x| \leq 1\}, \quad \int_{\mathbb{R}^d} \zeta(x)dx = 1, \quad \zeta(x) \geq 0.$$
Let $\varepsilon > 0$ and
$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right).$$
Show that
$$\|\zeta_\varepsilon \star f - f\|_{L^p(\mathbb{R}^d)} \leq C|\varepsilon|^d \|\nabla f\|_{L^p(\mathbb{R}^d)}.$$
4. *Show that* $\limsup_{\varepsilon \to 0} \sup_{\|f\|_{L^2} \leq 1} \|\zeta_\varepsilon \star f - f\|_{L^2} > 0$.
5. *Show that for all $s > 0$,* $\limsup_{\varepsilon \to 0} \sup_{\|f\|_{H^s} \leq 1} \|\zeta_\varepsilon \star f - f\|_{L^2} = 0$.

Exercice 3 (Weak convergence)**.** *Let H be a separable Hillbert space, V a dense subset of H. Let $u \in H$ and $(u_n)_{n \geq 1}$ be a sequence of elements in H. Show that*
$$u_n \rightharpoonup u \quad in \quad H$$
iff $(u_n)_{n \geq 1}$ is bounded in H and
$$\forall v \in V, \quad \lim_{n \to +\infty} \langle u_n, v\rangle_H = \langle u, v\rangle_H.$$

Exercice 4 (Separating bubbles)**.** *Let $d \geq 1$. Let $\ell \in \mathbb{N}^*$ and $(V_j)_{1 \leq j \leq \ell}$ be ℓ functions in $H^1(\mathbb{R}^d, \mathbb{C})$. Let $\mathbf{x}^j = (x_n^j)_{n \geq 1}$, $1 \leq j \leq \ell$ be ℓ sequences with*
$$\forall j \neq k, \quad |x_n^j - x_n^k| \to +\infty \quad as \quad n \to +\infty.$$

1. *Show that*
$$\|\sum_{j=1}^{\ell} V^j(\cdot - x_n^j)\|^2_{L^2} = \sum_{j=1}^{\ell} \|V^j\|^2_{L^2} + o(1) \quad as \quad n \to +\infty.$$
2. *Show that*
$$\|\sum_{j=1}^{\ell} \nabla V^j(\cdot - x_n^j)\|^2_{L^2} = \sum_{j=1}^{\ell} \|\nabla V^j\|^2_{L^2} + o(1) \quad as \quad n \to +\infty.$$

3. *Let $1 < p < +\infty$. Show that there exists a universal constant $C_{p,\ell} > 0$ such that for all complex numbers $(z_j)_{1\leq j\leq \ell}$,*
$$\left|\left|\sum_{j=1}^{\ell} z_j\right|^p - \sum_{j=1}^{\ell} |z_j|^p\right| \leq C_{p,\ell} \sum_{j\neq k} |z_j||z_k|^{p-1}.$$
4. *Let $1 < p < +\infty$. Show that if $V_j \in L^p(\mathbb{R}^d)$, then*
$$\|\sum_{j=1}^{\ell} V^j(\cdot - x_n^j)\|_{L^p}^p = \sum_{j=1}^{\ell} \|V^j\|_{L^p}^p + o(1) \quad \text{as} \quad n \to +\infty.$$

Exercice 5 (The trace map)**.** . *We define the trace map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{d-1})$ by*
$$\tau u(x') = u(0, x'), \qquad x' = (x_2, \dots, x_d).$$

1. *Show that for all $u \in \mathcal{S}(\mathbb{R}^d)$ and $\xi' \in \mathbb{R}^{d-1}$,*
$$\widehat{\tau u}(\xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{u}(\xi_1, \xi') d\xi_1.$$
2. *Show that for $s > 1/2$, $\exists C(s) > 0$ such that $\forall u \in \mathcal{S}(\mathbb{R}^d)$,*
$$\|\tau u\|_{H^{s-1/2}(\mathbb{R}^{d-1})} \leq C\|u\|_{H^s(\mathbb{R}^d)}.$$
Hint : use the previous question to derive the estimate
$$|\widehat{\tau u}(\xi')|^2 \leq \frac{1}{4\pi^2} \left(\int_{\mathbb{R}} |\widehat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi_1\right) \left(\int_{\mathbb{R}} \langle \xi \rangle^{-2s} d\xi_1\right)$$
and express $\int_{\mathbb{R}} \langle \xi \rangle^{-2s} d\xi_1$ in terms of $\langle \xi' \rangle$ (where we noted $\xi = (\xi_1, \xi')$).
3. *Let $s > 1/2$. Show that the trace application extends uniquely as a continuous map from $H^s(\mathbb{R}^d)$ onto $H^{s-1/2}(\mathbb{R}^{d-1})$.*
4. *Let $s > 1/2$ and $g \in H^{s-1/2}(\mathbb{R}^{d-1})$. Define*
$$\widehat{v}(\xi) = \widehat{g}(\xi') \frac{\langle \xi' \rangle^{2(s-1/2)}}{\langle \xi \rangle^{2s}}.$$
Show that $v \in H^s(\mathbb{R}^d)$ and $v(0, x') = Cg(x')$ for some constant $C \neq 0$. Conclude that the above trace map is surjective.

Exercice 6 (Space formulation of the homogeneous Sobolev norm)**.** . *Let $0 < s < 1$. Show that there exists $0 < c_1 < c_2$ such that for all $u \in H^s(\mathbb{R}^d)$, let*
$$I_s(u) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}} \, dx \, dy < \infty$$
then
$$c_1 \|u\|_{\dot{H}^s}^2 \leq I_s(u) \leq c_2 \|u\|_{\dot{H}^s}^2.$$
Hint : use Plancherel and Fubbini.

Exercice 7 (A commutator estimate)**.** *Let $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $s \in [0,1]$. Let the Fourier multiplier $\widehat{|D|^s v} \equiv |\xi|^s \widehat{v}$, and define the commutator*
$$A_s v = [|D|^s, \chi] \equiv |D|^s(\chi v) - \chi |D|^s v.$$

1. *Let $v \in \mathcal{D}(\mathbb{R}^d)$, compute $\widehat{A_s v}$ in the form of an integral operator ie $\widehat{A_s v}(\xi) = \int K(\xi, \xi') \hat{v}(\xi') d\xi'$.*
2. *Show that A_s is bounded on $L^2(\mathbb{R}^d)$.*