Introduction to non linear Analysis

Example sheet n^o 3 - NLS

Exercices to be done : 1-3

Exercice 1 (Cauchy problem for (NLS) in $\mathbb{R}$)**.** *Let $u_0 \in H^1(\mathbb{R})$. Show that there exists $T = T(\|u_0\|_{H^1})$ such that the map*

$$\Phi(u)(t,x) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}(u|u|^2)d\tau$$

is a contraction mapping on a suitable ball of the Banach space $E = L^\infty_{[0,T]}H^1_x$ equipped with the norm $\|u\|_E = \sup_{t\in[0,T]}\|u(t,\cdot)\|_{H^1}$.

Exercice 2 (A symmetry of the harmonic oscillator)**.** *We consider the cubic non linear harmonic oscillator in dimension 2*

$$i\partial_t u + \Delta u - |x|^2 u + u|u|^2 = 0.$$

1. *Define the renormalization*

$$\left| \begin{array}{l} u(t,x) = \frac{1}{L}w(s,y) \\ w(s,y) = e^{-i\frac{b(s)|y|^2}{4}}v(s,y) \\ y = \frac{x}{L}, \quad \frac{ds}{dt} = \frac{1}{L^2}. \end{array} \right.$$

   Show that $v(s,y)$ satisfies the same equation iff

$$\left| \begin{array}{l} \frac{L_s}{L} + b = 0 \\ \frac{b_s}{4} - \left(\frac{b^2}{4} + L^4\right) - \frac{b}{2}\frac{L_s}{L} = -1 \end{array} \right. \tag{0.1}$$

   where $f_s = \frac{df}{ds}$.

2. *Integrate the dynamical system* (0.1) *in time t. (hint : look for a conserved quantity and draw the phase portrait).*

Exercice 3 (An upper bound on blow up rate for (NLS))**.** *Let the focusing (NLS)*

$$\left| \begin{array}{l} i\partial_t u + \Delta u + u|u|^{p-1} = 0 \\ u(0,x) = u_0(x) \end{array} \right. , \quad x \in \mathbb{R}^2, \quad 3 < p < 5.$$

Let H^1_r be the set of H^1 functions with radial symmetry, then the Cauchy problem is well posed in H^1_r. We pick $u_0 \in H^1_r$ and assume that the solution blows up in finite time $0 < T < +\infty$. The aim of this problem is to derive an upper bound on $\|\nabla u(t)\|_{L^2}$ as $t \uparrow T$.
Integration by parts should be done without boundary terms (without justification). We let

$$s_c = 1 - \frac{2}{p-1}$$

and E_0 be the energy of the data. We recall Hölder :

$$|xy| \le \frac{1}{p}\left(\frac{|x|}{A}\right)^p + \frac{(A|y|)^{p'}}{p'}, \quad 1 \le p, p' \le +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad A > 0. \tag{0.2}$$

1. *Let $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ with spherical symmetry, prove the formulas :*

$$\frac{1}{2}\frac{d}{d\tau}\int \chi|u|^2 = Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right),$$

   and

$$\frac{1}{2}\frac{d}{d\tau}Im\left(\int \nabla\chi \cdot \nabla u\overline{u}\right) = \int \chi''|\nabla u|^2 - \frac{1}{4}\int \Delta^2\chi|u|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int \Delta\chi|u|^{p+1}.$$

2. *Prove that for all* $u \in H^1_r$,
$$\forall R > 0, \quad \|u\|^2_{L^\infty(r \geq R)} \leq \frac{2}{R}\|u\|_{L^2}\|\nabla u\|_{L^2}.$$
3. *Let* $R > 0$, $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ *with spherical symmetry and*
$$\psi(x) = \left| \begin{array}{l} \frac{|x|^2}{2} \quad pour \ \ |x| \leq 2 \\ 0 \quad pour \ \ |x| \geq 3 \end{array} \right. .$$
Let
$$\chi(x) = \psi_R(x) = R^2\psi\left(\frac{x}{R}\right),$$
show that
$$c(d,p)\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(d,p)\left[|E_0| + \int_{|x|\geq R}|u|^{p+1} + \frac{1}{R^2}\int_{2R\leq|x|\leq 3R}|u|^2\right]$$
for some constants $c(d,p), C(d,p) > 0$ *independant of R.*
4. *Prove using (0.2) that :*
$$\frac{c(d,p)}{2}\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\Im\left(\int \nabla\psi_R \cdot \nabla u\overline{u}\right) \leq C(u_0,d,p)\left[1 + \frac{1}{R^2} + \frac{1}{R^{\frac{2}{\alpha}}}\right] \tag{0.3}$$
with
$$\alpha = \frac{5-p}{p-1}.$$
5. *Integrate in time (0.3) and prove :* $\forall 0 < t_0 < t_2 < T$,
$$\int_{t_0}^{t_2}(t_2 - t)\|\nabla u(t)\|^2_{L^2}dt \leq C(u_0,d,p)\left[\frac{(t_2-t_0)^2}{R^{\frac{2}{\alpha}}} + R(t_2-t_0)\|\nabla u(t_0)\|_{L^2} + R^2\right].$$
6. *Choose* $R = (T-t_0)^{\frac{\alpha}{1+\alpha}}$ *and conclude that for t close enough to T :*
$$\int_{t_0}^{T}(T-t)\|\nabla u(t)\|^2_{L^2}dt \leq C(d,p,u_0)(T-t)^{\frac{\alpha}{2+\alpha}} + (T-t_0)^2\|\nabla u(t_0)\|^2_{L^2}.$$
7. *Show that for t close enough to T :*
$$\int_{t_0}^{T}(T-t)\|\nabla u(t)\|^2_{L^2}dt \leq C(d,p,u_0)(T-t)^{\frac{2\alpha}{1+\alpha}}.$$
8. *Conclude that there exists a sequence* $t_n \to T$ *such that*
$$\|\nabla u(t_n)\|_{L^2} \leq \frac{C(d,p,u_0)}{(T-t_n)^{\frac{1}{1+\alpha}}}.$$
(remark : this bound is sharp !).
9. *Open problem : prove any bound on blow up rate in the critical case* $p = 3$ *!*