Introduction to non linear Analysis

Example sheet n$^{\text{o}}$ 2- Fourier, tempered distributions and Sobolev spaces

Exercices to be done : 1-6-8-9-11.

Exercice 1. *Computation of Fourier transforms.*

1. *Compute the one dimensional Fourier transforms of the following functions*
$$\frac{1}{2}\mathbf{1}_{[-1,1]}(x), \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \frac{1}{2}e^{-|x|}, |x|e^{-|x|}, \frac{1}{\pi(x^2+1)}, \frac{\sin(x)}{\pi x}, \frac{1}{2i\pi(x+i)}.$$
2. *Let $f \in \mathcal{D}(\mathbb{R})$ and solve the ode $u'' - u = f$ using the variation of the constant method. Show that there is a unique solution with the boundary value $\lim_{x\to\pm\infty} u(x) = 0$ and give the corresponding representation formula in Greens' form :*
$$u(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.$$
3. *Let $f \in \mathcal{D}(\mathbb{R})$ and solve the ode $u'' - u = f$ in $\mathcal{S}'$ using the Fourier transform. Make the link with the Green formula above. Why is it that Fourier analysis does not see the homogeneous solution $f = 0, u(x) = e^x$?*

Exercice 2. *Fourier transform of Gaussians. Let A be a d-dimensional symmetric positive definite square matrix. Compute the Fourier transform of $f_A(X) = e^{-\frac{1}{2}\langle X, AX\rangle}$.*

Exercice 3. *Hilbert transform. Let $f \in L^2(\mathbb{R})$, we define its Hilbert transform by*
$$\widehat{Jf} = -\mathrm{i}\,\mathrm{sgn}(\xi)\widehat{f} \quad \textit{where} \quad \mathrm{sgn}(\xi) = \left| \begin{array}{l} 1 \;\; \textit{if} \;\; \xi > 0 \\ -1 \;\; \textit{if} \;\; \xi < 0. \end{array} \right.$$

1. *Show that J is an endomorphism of $L^2(\mathbb{R})$. Compute its norm.*
2. *Compute J^*.*
3. *Compute J^2 and conclude that J is invertible.*

Exercice 4. *Computation of Fourier transforms in $\mathcal{S}'(\mathbb{R})$. Computation the Fourier transform in $\mathcal{S}'(\mathbb{R})$ of the distributions*
$$1, \mathbf{1}_{x>0}, pv\left(\frac{1}{x}\right).$$

Exercice 5. *PDE in $\mathcal{S}'$. Solve in $\mathcal{S}'(\mathbb{R}^d)$:*
$$\Delta u + \sum_{j=1}^{n} x_j \partial_{x_j} u + du = 0.$$

Exercice 6. *Fundamental solution of Helmoltz.*

1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$ with spherical symmetry, show that its Fourier transform also has spherical symmetry*
2. *Let $\lambda > 0$ and $E : \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto \frac{e^{-\sqrt{\lambda}\|x\|}}{\|x\|}$, show that $E \in \mathcal{S}'(\mathbb{R}^3)$ and compute its Fourier transform.*
3. *Given $f \in \mathcal{S}(\mathbb{R}^3)$, solve the Helmoltz equation in $\mathcal{S}'(\mathbb{R}^3)$*
$$(\Delta - \lambda)u = f$$
and give the representation formula both in Fourier and space variables.

Exercice 7. *The trace map. We define the trace map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^{d-1})$ by*
$$\tau u(x') = u(0, x'), \qquad x' = (x_2, \dots, x_d).$$

1. *Show that for all* $u \in \mathcal{S}(\mathbb{R}^d)$ *and* $\xi' \in \mathbb{R}^{d-1}$,

$$\widehat{\tau u}(\xi') = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{u}(\xi_1, \xi')d\xi_1.$$

2. *Show that for* $s > 1/2$, $\exists C(s) > 0$ *such that* $\forall u \in \mathcal{S}(\mathbb{R}^d)$,

$$\|\tau u\|_{H^{s-1/2}(\mathbb{R}^{d-1})} \leq C\|u\|_{H^s(\mathbb{R}^d)}.$$

Hint : use the previous question to derive the estimate

$$|\widehat{\tau u}(\xi')|^2 \leq \frac{1}{4\pi^2}\left(\int_{\mathbb{R}} |\widehat{u}(\xi)|^2\langle\xi\rangle^{2s}d\xi_1\right)\left(\int_{\mathbb{R}}\langle\xi\rangle^{-2s}d\xi_1\right)$$

and express $\int_{\mathbb{R}}\langle\xi\rangle^{-2s}d\xi_1$ *in terms of* $\langle\xi'\rangle$ *(where we noted* $\xi = (\xi_1, \xi')$*).*

3. *Let* $s > 1/2$. *Show that the trace application extends uniquely as a continuous map from* $H^s(\mathbb{R}^d)$ *onto* $H^{s-1/2}(\mathbb{R}^{d-1})$.
4. *Let* $s > 1/2$ *and* $g \in H^{s-1/2}(\mathbb{R}^{d-1})$. *Define*

$$\widehat{v}(\xi) = \widehat{g}(\xi')\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}.$$

Show that $v \in H^s(\mathbb{R}^d)$ *and* $v(0, x') = Cg(x')$ *for some constant* $C \neq 0$. *Conclude that the above trace map is surjective.*

Exercice 8. *Uniform regularization in Sobolev spaces. Let* $\zeta \in D(\mathbb{R}^d)$ *with*

$$Supp(\zeta) \subset \{|x| \leq 1\}, \quad \int_{\mathbb{R}^d}\zeta(x)dx = 1, \quad \zeta(x) \geq 0.$$

For $\varepsilon > 0$, *we define* $\zeta_\varepsilon(x) = \frac{1}{\varepsilon^d}\zeta\left(\frac{x}{\varepsilon}\right)$.

1. *Use Fourier analysis to prove that* $\forall f \in L^2(\mathbb{R}^d)$, $\|\zeta_\varepsilon \star f - f\|_{L^2(\mathbb{R}^d)} \to 0$.
2. *Let* $s > 0$. *Show that* $\lim_{\varepsilon\to 0}\sup_{\|f\|_{H^s(\mathbb{R}^d)}\leq 1}\|\zeta_\varepsilon \star f - f\|_{L^2(\mathbb{R}^d)} = 0$.
3. *Show that* $\sup_{\|f\|_{L^2(\mathbb{R}^d)}\leq 1}\|\zeta_\varepsilon \star f - f\|_{L^2(\mathbb{R}^d)}$ *does not converge to zero as* $\varepsilon \to 0$.

Exercice 9. *Space formulation of the homogeneous Sobolev norm. Let* $0 < s < 1$. *Show that there exists* $0 < c_1 < c_2$ *such that for all* $u \in H^s(\mathbb{R}^d)$, *let*

$$I_s(u) = \int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u(x+y) - u(x)|^2}{|y|^{d+2s}}\,dx\,dy < \infty$$

then

$$c_1\|u\|^2_{\dot{H}^s} \leq I_s(u) \leq c_2\|u\|^2_{\dot{H}^s}.$$

Hint : use Plancherel and Fubbini.

Exercice 10. *Let* $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ *and* $s \in [0, 1]$. *Let the Fourier multiplier* $\widehat{|D|^s v} \equiv |\xi|^s\widehat{v}$. *and define the commutator*

$$A_s v = [|D|^s, \chi] \equiv |D|^s(\chi v) - \chi|D|^s v.$$

1. *Let* $v \in \mathcal{D}(\mathbb{R}^d)$, *compute* $\widehat{A_s v}$ *in the form of an integral operator ie* $\widehat{A_s v}(\xi) = \int K(\xi, \xi')\hat{v}(\xi')d\xi'$.
2. *Show that* A_s *is bounded on* $L^2(\mathbb{R}^d)$.

Exercice 11. *Let* $s > \frac{d}{2}$. *Show that* $\exists c > 0$ *such that*

$$\forall u \in H^s(\mathbb{R}^d), \quad \|u\|_{L^\infty(\mathbb{R}^d)} \leq c\|u\|_{L^2}^{1-\frac{d}{2s}}\|u\|_{\dot{H}^s}^{\frac{d}{2s}}.$$