Introduction to non linear Analysis

Example sheet n^o 1- Distributions

Exercices to be done : 1-3-4-6-7

Exercice 1. *Division by x in $\mathcal{D}'(\mathbb{R})$.*

1. *Solve $xT = 0$ in $\mathcal{D}'(\mathbb{R})$. More generally, solve $x^m T = 0$, $m \in \mathbb{N}$, in $\mathcal{D}'(\mathbb{R})$.*
2. *Given $S \in \mathcal{D}'(\mathbb{R})$, solve $xT = S$ in $\mathcal{D}'(\mathbb{R})$.*

Exercice 2. *ODE in $\mathcal{D}'(\mathbb{R})$.*

1. *Let $T \in \mathcal{D}'(\mathbb{R})$ with $T' = 0$ in $\mathcal{D}(\mathbb{R})$. Show that T is a constant.*
2. *Solve $T' - T = \delta$ in $\mathcal{D}'(\mathbb{R})$.*

Exercice 3. *Limit of distributions.*

1. *Show that the linear form on $\mathcal{D}(\mathbb{R})$ given by $\left\langle \mathrm{pv}\left(\frac{1}{x}\right), \phi)\right\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\phi(x)}{x} dx$ belongs to $\mathcal{D}'$.*
2. *Given $\varepsilon > 0$, let the complex valued function $f_\varepsilon(x) = \frac{1}{x+i\varepsilon}$, compute $\lim_{\varepsilon\to 0} f_\varepsilon$ in $\mathcal{D}'(\mathbb{R})$.*

Exercice 4. *Derivative and translations. Let $\phi \in \mathcal{D}(\mathbb{R})$, $h \in \mathbb{R}$, we define the translation operation by $\tau_h\phi(x) = \phi(x+h)$. Let $T \in \mathcal{D}'(\mathbb{R})$, we define the translation operation by $\langle \tau_h T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \tau_{-h}\phi\rangle_{\mathcal{D}',\mathcal{D}}$. Show that*

$$\lim_{h\to 0} \frac{\tau_h T - T}{h} = T' \quad in \quad \mathcal{D}'(\mathbb{R}).$$

Exercice 5. *Computing derivatives in $\mathcal{D}'(\mathbb{R}^d)$.*

1. *Let the Heaviside function be $H(x) = \mathbf{1}_{x>0}$. Let $\tilde{H}(x_1, ..., x_N) = H(x_1)...H(x_N)$ and $\alpha = (1, ...1)$. Show that $\partial^\alpha \tilde{H} = \delta_0$.*
2. *Show that the linear form on $\mathcal{D}(\mathbb{R}^2)$ given by $\langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \int_{\mathbb{R}} \phi(x, x)dx$ defines an element of $\mathcal{D}'(\mathbb{R}^2)$. Compute $\partial_x T + \partial_y T$.*

Exercice 6. *Distributions with support a singleton. Let $T \in \mathcal{D}'(\mathbb{R})$ with finite order $p \in \mathbb{N}$ such that*

$$\forall \phi \in \mathcal{D}(\mathbb{R}\backslash\{0\}), \quad \langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = 0,$$

we want to show that $T = \sum_{i=0}^p c_i \frac{d^i}{dx^i} \delta_{x=0}$.

1. *Let $\chi \in \mathcal{D}(\mathbb{R})$ with $\chi(x) = 1$ for $|x| \leq 1$ and $\mathrm{Supp}(\chi) \subset [-2, 2]$. Let $\chi_\varepsilon(x) = \chi\left(\frac{x}{\varepsilon}\right)$. Let $\phi \in \mathcal{D}(\mathbb{R})$, show that $\langle T, \phi\rangle_{\mathcal{D}',\mathcal{D}} = \langle T, \chi_\varepsilon\phi\rangle_{\mathcal{D}',\mathcal{D}}$.*
2. *Assume $\frac{d^i\phi}{dx^i} = 0$ for $0 \leq i \leq p$. Show that $\lim_{\varepsilon\to 0}\langle T, \chi_\varepsilon\phi\rangle_{\mathcal{D}',\mathcal{D}} = 0$ and conclude.*
3. *Extend the result to $\mathcal{D}'(\mathbb{R}^d)$.*

Exercice 7. *Fundamental solution of the Laplacian.*

1. *Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with radial symmetry (ie $\phi(x) \equiv \phi(r)$ with $r = \sqrt{\sum_{i=1}^d x_i^2}$.) Show that*

$$\Delta\phi \equiv \sum_{i=1}^d \frac{\partial^2\phi}{\partial x_i^2} = \frac{d^2\phi}{dr^2} + \frac{d-1}{r}\frac{d\phi}{dr}.$$

2. *Let $x \in \mathbb{R}^d$ and define*

$$E_d(x) = \begin{cases} |x|^{-d-2} & \textit{if } d \geq 3, \\ \ln|x| & \textit{if } d = 2. \end{cases}$$

   Show that $E_d \in \mathcal{C}^\infty(\mathbb{R}^d\backslash\{0\})$ with $\Delta E_d = 0$ in $\mathcal{D}'(\mathbb{R}^d\backslash\{0\})$.
3. *Let $\phi \in \mathcal{D}(\mathbb{R}^d)$. Show that*

$$\langle \Delta E_d, \varphi\rangle = \lim_{\varepsilon\to 0^+} \int_{\|x\|>\varepsilon} E_d \Delta\varphi \, dx.$$

4. *Let $d = 2, 3$. By transforming the above integral using Green's formula, compute ΔE_d in $\mathcal{D}'(\mathbb{R}^d)$.*