Part III Commutative Algebra

Example Sheet III, 2021

Note: If you would like to receive feedback, please turn in solutions to Questions 2, 5, and 9 by noon on November 26th, at which time solutions will be posted. You turn your work in via the moodle course page.

1. Let $S = k[x_0, \ldots, x_n]$ be the polynomial ring viewed in the standard way as a graded ring.

   (a) Let $f \in S_d$ be a homogeneous polynomial of degree d. Use the exact sequence

   $$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow S/(f) \longrightarrow 0$$

   (where $\cdot f$ denotes multiplication by f) to compute the Hilbert polynomial of the graded S-module $S/(f)$. [For those who have studied algebraic geometry, consider the case when $n = 2$. Does the constant term of the Hilbert polynomial mean anything to you?]

   (b) Let $f \in S_d$, $g \in S_e$ be two homogeneous polynomials which are coprime. Using a similar idea as to (a), calculate the Hilbert polynomial of $S/(f, g)$.

2. Continue with the notation as in Question 1. Let $I \subseteq S$ be a homogeneous ideal, and let n be the degree of the Hilbert polyomial $f_{S/I}(x)$ of the graded S-module S/I. We define the *degree* of I to be the coefficient of x^n in $f_{S/I}(x)$ times $n!$. [Actually, this is the degree of the projective variety defined by I, but I'm trying to avoid this language.]

   (a) For $I = (f)$ or $I = (f, g)$ as in Question 1, calculate the degree of I.

   (b) Let I be of degree d, and let $f \in S$ be a homogeneous polynomial of degree e such that f is not a zero-divisor in S/I. Show that the degree of $I + (f)$ is e times the degree of I. [This is known as Bézout's theorem.]

3. Let A be a ring, $\mathfrak{p} \subseteq A$ a prime ideal. Show that $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ is the field of fractions of $A/\mathfrak{p}$. Thus in particular, if $\mathfrak{m} \subseteq A$ is maximal, we see $A/\mathfrak{m} \cong A_\mathfrak{m}/\mathfrak{m}A_\mathfrak{m}$. Now show that $\mathfrak{m}^k/\mathfrak{m}^{k+1} \cong (\mathfrak{m}A_\mathfrak{m})^k/(\mathfrak{m}A_\mathfrak{m})^{k+1}$. [Note: we have implicitly used this in lecture already, so I thought it would be a good idea to state explicitly.]

4. Let A be a Noetherian ring, $I \subseteq A$ an ideal, $\mathrm{gr}_I(A) := \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$ the graded ring defined using I.

   (a) Show $\mathrm{gr}_I(A)$ is Noetherian.

   (b) Suppose further that A is a local ring, $I = \mathfrak{m}$ the unique maximal ideal. Show that if $\mathrm{gr}_\mathfrak{m}(A)$ is an integral domain, so is A.

5. *Jacobian criterion for regularity.* Let $I = (f_1, \ldots, f_m) \subseteq k[x_1, \ldots, x_n]$ be an ideal contained in the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. Let

   $$A = (k[x_1, \ldots, x_n]/I)_{\bar{\mathfrak{m}}},$$

   where $\bar{\mathfrak{m}} = \mathfrak{m}/I$. Show that A is a regular local ring if and only if the *Jacobian matrix*

   $$((\partial f_i/\partial x_j)|_{x_1=\cdots=x_n=0})_{1\le i\le m, 1\le j\le n}$$

   has rank $n - d$, where $d = \dim A$.

6. We will develop the theory of *completions* in the following sequence of exercises. This is important material, but there is insufficient time to cover in lectures. While this material is covered, say, in Chapter 10 of Atiyah and MacDonald, try to do this series of problems on your own.

   We start with a special case of *inverse limits* (which are themselves a special case of the notion of *limit* in category theory). Suppose given a collection G_n, $n \ge 1$ of abelian groups equipped with homomorphisms $\phi_n : G_{n+1} \to G_n$ for each $n \ge 1$. We call such data an *inverse system*.

   Then the *inverse limit* of $\{G_n\}$ is an abelian group G equipped with maps $g_n : G \to G_n$ for each n and with $\phi_n \circ g_{n+1} = g_n$, satisfying the following universal property:

   Given any abelian group H and homomorphisms $h_n : H \to G_n$ with $\phi_n \circ h_{n+1} = h_n$, there exists a unique homomorphism $h : H \to G$ with $h_n = g_n \circ h$ for all n. In other words we have, for all n, a commutative diagram

$$\begin{tikzcd}
& & G_{n+1} \arrow[dd, "\phi_n"] \\
H \arrow[r, "h"] \arrow[urr, bend left, "h_{n+1}"] \arrow[drr, bend right, "h_n"'] & G \arrow[ur, "g_{n+1}"'] \arrow[dr, "g_n"] & \\
& & G_n
\end{tikzcd}$$

Show that the inverse limit exists. Show further that if G_n has the structure of a ring (resp. A-module) and the ϕ_n are ring homomorphisms (resp. A-module homomorphisms), then the inverse limit is also a ring or A-module.

We write

$$\lim_{\longleftarrow} G_n$$

for the inverse limit.

7. Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ be three inverse systems of abelian groups as in Question 6, forming an exact sequence in the sense that for each n there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d] & B_{n+1} \arrow[r] \arrow[d] & C_{n+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are the maps of the three inverse systems and the rows are short exact sequences. Show there is a short exact sequence

$$0 \longrightarrow \lim_{\leftarrow} A_n \longrightarrow \lim_{\leftarrow} B_n \longrightarrow \lim_{\leftarrow} C_n.$$

Show furthermore that the right-most map is surjective if the inverse system $\{A_n\}$ has all maps $\phi_n : A_{n+1} \to A_n$ surjective.

8. A frequent situation is that an inverse system arises from a filtration, i.e., $G \supset G_1 \supset G_2 \supset \cdots$, so that there are natural surjections $\phi_n : G/G_{n+1} \to G/G_n$, hence giving

$$\lim_{\longleftarrow} G/G_n.$$

Here we give a topological interpretation. First, we may put a topology on G with basis of open sets given by $\{g + G_n \,|\, g \in G, n \geq 1\}$.

Show that this topology is Hausdorff if $\bigcap_{n=1}^{\infty} G_n = 0$.

We may define the completion of G with respect to this topology in the usual way. A sequence g_n in G is *Cauchy* if for any open neighbourhood $U \subseteq G$ of 0, there exists an N such that for all $n, n' \geq N$, $g_n - g_{n'} \in U$. Two Cauchy sequences g_n, g'_n are *equivalent* if for each open neighbourhood U of 0, there exists an N such that for all $n \geq N$, $g_n - g'_n \in U$. The completion $\widehat{G}$ of G is then defined to be the set of Cauchy sequences of G modulo equivalence.

Show that $\widehat{G}$ has the structure of an abelian group, and there is a canonical map $G \to \widehat{G}$ with kernel $\bigcap_{n=1}^{\infty} G_n$.

Finally, show there is an isomorphism of abelian groups

$$\lim_{\longleftarrow} G/G_n \cong \widehat{G},$$

thus showing the inverse limit does not depend on the filtration but only on the induced topology.

9. Given a ring A and an ideal $I \subseteq A$, we obtain an inverse system with $A_n = A/I^n$ and $\phi_n : A/I^{n+1} \to A/I^n$ the obvious surjection. We write $\widehat{A} := \lim_{\leftarrow} A_n$, but bear in mind this depends on I. We call $\widehat{A}$ the *I-adic completion* of A.

Similarly, if M is an A-module, we obtain an inverse system of modules $M_n := M/I^n M$, and write $\widehat{M}$ for the inverse limit of the M_n, the *I-adic completion* of M.

(a) Describe the ring $k[[x_1, \ldots, x_n]]$, the completion of $k[x_1, \ldots, x_n]$ with respect to the maximal ideal $(x_1, \ldots, x_n)$. Describe the ring $\mathbb{Z}_p$, the completion of the integers $\mathbb{Z}$. [The former ring is called the *ring of formal power series* and the latter ring is called the ring of p-adic numbers.]

(b) Show that if A is Noetherian and

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of finitely generated A-modules, then

$$0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$$

is an exact sequence of $\widehat{A}$-modules.

(c) Show there is a canonical homomorphism $M \to \widehat{M}$, and show the kernel of this homomorphism is $\bigcap_{n=0}^{\infty} I^n M$.

(d) Show for M an A-module, there is a homomorphism $\widehat{A} \otimes_A M \to \widehat{M}$ given as a composition

$$\widehat{A} \otimes_A M \to \widehat{A} \otimes_A \widehat{M} \to \widehat{A} \otimes_{\widehat{A}} \widehat{M} \cong \widehat{M}.$$

Show that if M is finitely generated, then this homomorphism is surjective. If in addition A is Noetherian, this homomorphism is an isomorphism. [We remark this shows that $\widehat{A}$ is a flat A-algebra. This follows from the fact that flatness of an A-module N can be tested on injective morphisms of finitely generated A-modules.]

10. Let A be Noetherian and $\widehat{A}$ its I-adic completion. Show the following:

    (a) $\widehat{I} = I^e \cong \widehat{A} \otimes_A I$, where I^e is the extension of I under the canonical homomorphism $A \to \widehat{A}$.

    (b) $\widehat{I^n} = (\widehat{I})^n$.

    (c) $I^n/I^{n+1} \cong \widehat{I}^n/\widehat{I}^{n+1}$.

    (d) $\widehat{I}$ is contained in the Jacobson radical of $\widehat{A}$.

    Use this to show that if A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{m}$-adic completion $\widehat{A}$, then $\widehat{A}$ is a local ring with maximal ideal $\widehat{\mathfrak{m}}$.

11. Let A be a local ring with maximal ideal $\mathfrak{m}$, and $\widehat{A}$ its completion. For any $f(x) \in \widehat{A}[x]$, write $\bar{f}(x) \in (A/\mathfrak{m})[x]$ for its reduction. Suppose that $f(x)$ is monic of degree n, and suppose there exists $\bar{a} \in A/\mathfrak{m}$ such that $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$. Show there exists an $a \in \widehat{A}$ with $f(a) = 0$. [This is known as Hensel's Lemma.]

12. (a) Show that $k[x, y]/(y^2 - x^2(1 + x))$ is an integral domain but that $k[[x, y]]/(y^2 - x^2(1 + x))$ is not.

    (b) Let $A = k[[x, y]]/(x^2 - y^3)$, $\mathfrak{m} = (x, y)$ the unique maximal ideal. Show that $\mathrm{gr}_{\mathfrak{m}}(A)$ is not an integral domain. [A is itself an integral domain, but you don't need to show this. This follows as usual from the fact that $k[[x, y]]$ is a UFD and $x^2 - y^3$ is irreducible.]