Part III Commutative Algebra

Example Sheet II, 2021

Note: If you would like to receive feedback, please turn in solutions to Questions 4, 7, and 9 by noon on November 12th, at which time solutions will be posted. You turn your work in via the moodle course page.

1. *Geometric interpretation of primary decomposition.* Let X be a topological space. A non-empty subset $Z \subseteq X$ is *irreducible* if whenever $Z = Z_1 \cup Z_2$ with Z_1, Z_2 closed subsets of Z, we have either $Z = Z_1$ or $Z = Z_2$.

   A topological space is *Noetherian* if it satisfies the descending chain condition for closed subsets.

   Show that if $Z \subseteq X$ is a closed subset with X a Noetherian topological space, then there is a decomposition
   $$Z = Z_1 \cup \cdots \cup Z_r$$
   with the Z_i irreducible closed subsets. Show this decomposition is unique (up to reordering) if it is irredundant, i.e., we don't have $Z_i \subseteq Z_j$ for any j.

   Now let A be a ring, $I \subseteq A$ an ideal. Show that I prime implies that $V(I)$ is irreducible.

   Show that if A is Noetherian, then $\operatorname{Spec} A$ is a Noetherian topological space. Show that if A is Noetherian, $Z = V(I)$ and $Z = Z_1 \cup \cdots \cup Z_r$ is an irredundant irreducible decomposition, then there is a one-to-one correspondence between the set $\{Z_i\}$ and the minimal associated primes of A/I.

2. Let $I = (x^2, xy, xz, yz) \subseteq k[x, y, z]$. Find the associated primes of $k[x, y, z]/I$ and a primary decomposition for I.

3. Show the following statements about the support of an A-module:

   (a) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, then $\operatorname{Supp}(M_2) = \operatorname{Supp}(M_1) \cup \operatorname{Supp}(M_3)$.

   (b) If $M = \sum_i M_i$ for submodules $M_i \subseteq M$, then $\operatorname{Supp}(M) = \bigcup \operatorname{Supp}(M_i)$.

   (c) For $\mathfrak{p} \subset A$ a prime ideal, write $k(\mathfrak{p}) := A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. If M is finitely generated, show that $\mathfrak{p} \in \operatorname{Supp}(M)$ if and only if $M \otimes_A k(\mathfrak{p}) \neq 0$.

   (d) If M, N are finitely generated A-modules, then $\operatorname{Supp}(M \otimes_A N) = \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$.

   (e) If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $\operatorname{Supp}(B \otimes_A M) = (f^*)^{-1}(\operatorname{Supp}(M))$.

   [Note: You may find useful the identity on tensor products $(M \otimes_A B) \otimes_B C \cong M \otimes_A C$ given an A-module M, an A-algebra B and a B-algebra C.]

4. Let A be a ring and let $A[x]$ be the polynomial ring. If $I \subseteq A$ is an ideal, denote by $I[x]$ the set of all polynomials in $A[x]$ with coefficients in I.

   (a) Show $I[x]$ is the extension I^e of I to $A[x]$.

   (b) If $\mathfrak{p}$ is a prime ideal of A, show that $\mathfrak{p}[x]$ is a prime ideal of $A[x]$.

   (c) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A, then $\mathfrak{q}[x]$ is a $\mathfrak{p}[x]$-primary ideal in $A[x]$.

   (d) If $I = \bigcap_{i=1}^n \mathfrak{q}_i$ is an irredundant primary decomposition of I in A, then $I[x] = \bigcap_{i=1}^n \mathfrak{q}_i[x]$ is an irredundant primary decomposition in $A[x]$.

   (e) If $\mathfrak{p}$ is a minimal prime of $V(I)$, then $\mathfrak{p}[x]$ is a minimal prime of $V(I[x])$.

5. In a ring A, let $D(A)$ denote the set of prime ideals $\mathfrak{p}$ which satisfy the following condition: there exists an $a \in A$ such that $\mathfrak{p}$ is minimal in $V(\operatorname{Ann}(a))$.

   (a) Show that x is a zero divisor in A if and only if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in D(A)$.

   (b) Let S be a multiplicatively closed subset of A. Under the canonical identification of $\operatorname{Spec} S^{-1}A$ with a subset of $\operatorname{Spec} A$, show that
   $$D(S^{-1}A) = D(A) \cap \operatorname{Spec} S^{-1}A.$$

   (c) If the 0 ideal has a primary decomposition, show that $D(A)$ is the set of associated primes of A. [Note: While results in lecture were always proved for Noetherian rings, the argument that if $N \subseteq M$ has a primary decomposition, then the associated primes of M/N are precisely the primes appearing in an irredundant primary decomposition of N is easily checked to apply without any hypotheses.]

[Note: We are not assuming A is Noetherian; otherwise problem this would follow essentially immediately from results in lecture.]

6. For $\mathfrak{p} \subset A$ a prime ideal, denote by $S_{\mathfrak{p}}(0)$ the kernel of the canonical homomorphism $A \to A_{\mathfrak{p}}$. Prove:

   (a) $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.

   (b) If $\mathfrak{p} \supseteq \mathfrak{p}'$ then $S_{\mathfrak{p}}(0) \subseteq S_{\mathfrak{p}'}(0)$.

   (c) $\sqrt{S_{\mathfrak{p}}(0)} = \mathfrak{p}$ if and only if $\mathfrak{p}$ is a minimal prime ideal of A.

   (d) $\bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0) = 0$, where $D(A)$ is the set defined in the previous exercise..

   [Note if there are embedded primes, this doesn't give a primary decomposition, and at any rate, $D(A)$ may not be finite.]

7. Let A be a ring such that (1) for every maximal ideal $\mathfrak{m}$ of A, the local ring $A_{\mathfrak{m}}$ is Noetherian; (2) for each non-zero $x \in A$, the set of maximal ideals of A which contain x is finite.

   Show that A is Noetherian.

   [Hint: You may use the theorem that a ring A is Noetherian if and only if every ideal of A is finitely generated. Start with an ideal I, and try to find a finite set of generators by localizing at various maximal ideals.]

8. Let k be a field and let $A = k[x_1, x_2, x_3, \ldots]$ be the polynomial ring in countably many variables. Let $m_1, m_2 \ldots$ be an increasing sequence of positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, \ldots, x_{m_{i+1}})$ and let S be the complement of $\bigcup_{i=1}^{\infty} \mathfrak{p}_i$.

   (a) Show that S is multiplicatively closed.

   (b) Show that the maximal ideals of $S^{-1}A$ are the localizations $S^{-1}\mathfrak{p}_i$. [Hint: This would be easy if S was a union of only a finite number of primes. First understand that case and then see how the argument can be generalized. You need to use the specific form of the ideals $\mathfrak{p}_i$ and the fact we are dealing with polynomials, not elements of an arbitrary ring! I think this is hard.]

   (c) Using Q7, show that $S^{-1}A$ is Noetherian.

   (d) Show that the height of $S^{-1}\mathfrak{p}_i$ is $m_{i+1} - m_i$, and hence $\dim S^{-1}A = \infty$.

   This is a famous example of Nagata.

9. Let A be a ring (not necessarily Noetherian). Show that

$$1 + \dim A \leq \dim A[x] \leq 1 + 2 \dim A.$$

   [Hint: consider the obvious inclusion $f : A \to A[x]$ and the induced map $f^* : \operatorname{Spec} A[x] \to \operatorname{Spec} A$. For $\mathfrak{p} \in \operatorname{Spec} A$, make use of Example Sheet I, Q12 (d) to identify $(f^*)^{-1}(\mathfrak{p})$ with $\operatorname{Spec} k(\mathfrak{p})[x]$ and use the fact that $\dim k[x] = 1$ when k is a field.]

   [Remark: In fact, for any n and $n \leq m \leq 1 + 2m$, one can find a (necessarily non-Noetherian) domain A with $\dim A = n$ and $\dim A[x] = m$.]