III Commutative Algebra   Michaelmas Term 2020

EXAMPLE SHEET 4

All rings R are commutative with a $1 \neq 0$.

1. (i) Let M be an R-module and I and ideal of R. Suppose that $M_P = 0$ for all maximal ideals P containing I. Show that $M = IM$.

(ii) Now assume R is noetherian and M is f.g. as an R-module. Using Krull's theorem and (i), show that

$$\bigcap_{n=1}^{\infty} I^n M = \bigcap_P \ker(M \to M_P)$$

where P runs over all maximal ideals containing I.

(iii) Deduce that $\hat{M} = 0$ iff $\mathrm{Supp}(M) \cap V(I) = \emptyset$ (in $\mathrm{Spec}(R)$).

[Recall that the *support* of M is defined to be the set $\mathrm{Supp}(M)$ of prime ideals P of R such that $M_P \neq 0$. Properties are explored on Sheet 2 Q17.]

2. Let R be noetherian, I an ideal of R and $\hat{R}$ the I-adic completion. For any $r \in R$, let $\hat{r}$ be its image in $\hat{R}$. Show that r is not a zero-divisor in R iff $\hat{r}$ is not a zero-divisor in $\hat{R}$. Does this imply that if R is an I.D. then $\hat{R}$ is an I.D.?

3. Let R be a noetherian local ring, P its maximal ideal and k its residue field. Let M be a f.g. R-module. Show that TFAE:

(i) M is free;

(ii) M is flat;

(iii) the mapping of $P \otimes M$ into $R \otimes M$ is injective;

(iv) $\mathrm{Tor}_1^k(k, M) = 0$.

4. If M is an R-module, show that TFAE

(i) M is flat

(ii) $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and all R-modules N.

(iii) $\mathrm{Tor}_1^R(M, N) = 0$ for all R-modules N.

5. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with N'' flat. Show that N' is flat iff N is flat.

6. Let N be an R-module. Show that N is flat iff $\mathrm{Tor}_1(R/I, N) = 0$ for all finitely-generated ideals I of R.

7. Let R, S be rings (not necessarily noetherian), S a multiplicatively closed subset of R, and $\varphi : R \to S^{-1}R$ the canonical homomorphism.

(i) Show that the induced map $\varphi^* : \mathrm{Spec}(S^{-1}R) \to X =$Spec$(R)$ is a homeomorphism of Spec$(S^{-1}R)$ onto its image in X. Denote this image by $S^{-1}X$.

(ii) Let $f : R \to R'$ be a ring homomorphism. Let $Y =$Spec(R'). Let $f^* : Y \to X$ be the mapping associated to f. Identifying Spec$(S^{-1}R)$ with its canonical image $S^{-1}X$ in X, and Spec$(S^{-1}R')$ (= Spec$(f(S)^{-1}R')$) with its canonical image $S^{-1}Y$ in Y, show that $S^{-1}f^*$:Spec$(S^{-1}R')$ $\to$Spec$(S^{-1}R)$ is the restriction of f^* to $S^{-1}Y$, and that $S^{-1}Y = {f^*}^{-1}(S^{-1}X)$.

(iii) Let I be an ideal of R and let J be its extension in R' (meaning it's the ideal $R'f(I)$ generated by $f(I)$ in R'). Let $g : R/I \to R'/J$ be the homomorphism induced by f. Identifying Spec(R/I) with its canonical image $V(I)$ in X, and Spec(R'/J) with its image $V(J)$ in Y, show that g^* is the restriction of f^* to $V(J)$.

(iv) Let P be a prime ideal of R. Take $S = R \setminus P$ in (ii) and then reduce mod $S^{-1}P$ as in (iii). Deduce that the subspace ${f^*}^{-1}(P)$ of Y is naturally homeomorphic to Spec $(R'_P/PR'_P) =$ Spec$(k(P) \otimes_R R')$, where $k(P)$ is the residue field of the local ring R_P.

[Spec$(k(P) \otimes_R R')$ is called the *fiber* of f^* over P.]

(v) Deduce from (iv) that

$$1 + \dim R \leq \dim R[X] \leq 1 + 2 \dim R.$$

[Hint: you may need the easy fact that if P is a prime ideal of R, then $P[X]$, the set of all polynomials in $R[X]$ with coefficients in P, is a prime ideal in $R[X]$.]

8. Let R be a noetherian ring. Show that

$$\dim R[X] = 1 + \dim R,$$

and hence, by induction on n, that

$$\dim R[X_1, \ldots, X_n] = n + \dim R.$$

9. (i) If $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ are short exact sequences of R-modules and P, P' are projective, then show $K \oplus P'$ is isomorphic to $K' \oplus P$. This is called *Schanuel's lemma* after Stephen Schanuel who proved it during a class given by Irv Kaplansky.

(ii) Prove a mild extension extension for the R-module M. Namely, given two exact sequences

$$0 \to K \to P_n \to P_{n-1} \to \cdots \to P_1 \to M \to 0$$

$$0 \to L \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to M \to 0,$$

where the Ps and Qs are projective. Then

$$K \oplus Q_n \oplus P_{n-1} \oplus \cdots \cong L \oplus P_n \oplus Q_{n-1} \oplus \cdots$$

where, if n is odd, the direct sums terminate in Q_1 and P_1, respectively; if n is even, they terminate in P_1 and Q_1.

10. Assume R is noetherian and I an ideal. Show that I is contained in the Jacobson radical of R iff every maximal ideal of R is closed for the I-adic topology. [A noetherian topological ring in which the topology is defined by an ideal containing in $J(R)$ is called a *Zariski ring*. Examples are local rings and by Q.13(iv) below, I-adic completions.]

11. Show that the additive group of an R-module M is a topological abelian group with respect to the I-adic topology. (You have to show that the maps $M \times M \to M \quad (x, y) \mapsto x + y$ and $M \to M \quad m \mapsto -m$ are continuous.)

12. (i) Let A be an integral domain. An A-module D is said to be *divisible* if for every $d \in D$ and every non-zero $r \in A$ there exists $c \in D$ such that $rc = d$. Note that we do not require the uniqueness of c. Assuming that R is a principal ideal domain show that an A-module is injective iff it is divisible.

(ii) Find an example of a ring and module for this ring which is divisible but not injective.

13. Let R be noetherian, $\hat{R}$ its I-adic completion. Show that

(i) $\hat{I} = \hat{R}I \cong \hat{R} \otimes_R I$;

(ii) $\widehat{(I^n)} = (\hat{I})^n$;

(iii) $I^n/I^{n+1} \cong \hat{I}^n/\hat{I}^{n+1}$;

(iv) the ideal $\hat{I}$ is contained in the Jacobson radical of $\hat{R}$.

Deduce that if (noetherian) R is a local ring with maximal ideal P, then the I-adic completion $\hat{R}$ of R is a local ring with maximal ideal $\hat{P}$.

14. Let R be a local ring, P its maximal ideal. Assume that R is P-adically complete. For any polynomial $p(X) \in R[X]$, let $\bar{p}(X) \in (R/P)[X]$ denote its reduction mod P. Suppose that $p(X)$ is monic of degree n; suppose also that there exist coprime monic polynomials $\bar{q}(X), \bar{r}(X) \in (R/P)[X]$ of degrees $d, n-d$ with $\bar{p}(X) = \bar{q}(X)\bar{r}(X)$. Show that we can lift $\bar{q}(X), \bar{r}(X)$ back to monic polynomials $q(X), r(X) \in R[X]$ such that $p(X) = q(X)r(X)$. This is known as *Hensel's lemma*.

15. Let R be a (not necessarily commutative) ring. Show that every left R-module is isomorphic to a submodule of an injective module. [Hint: Prove this result first in the case of abelian groups, i.e. for $R = Z$. Then if S is any ring and M is injective as Z-module, show that $\mathrm{Hom}_Z(S, M)$ is an injective S-module. Then deduce any S-module can be embedded in an injective S-module.]

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