III **Commutative Algebra** **Michaelmas Term 2020**

EXAMPLE SHEET 3

All rings R are commutative with a 1. A usually denotes a domain.

1. A chain of prime ideals is *maximal* if it is not a proper subset of another chain of primes. Let R be a ring of finite dimension in which all maximal chains of prime ideals have the same length and let P be a prime ideal of R. Show that:

(i) The quotient R/P is also a ring of finite dimension in which all maximal chains of prime ideals have the same length.

(ii) $\dim R = \dim R/P + \text{codim } P$.

(iii) $\dim R_P = \dim R$, if P is maximal.

2. Let K be a field and let $n \in N$. We know that $\dim K[X_1, \ldots, X_n] = n$. Show by induction on n that all maximal chains of prime ideals in $K[X_1, \ldots, X_n]$ have length n. If R is a f.g. algebra over K, and R is an I.D., show that every maximal chain of prime ideals in R has length $\dim R$.

3. Let A be a f.g. algebra over a field. Assume A is an I.D. of dimension d. Let $r \in A$ be neither 0 nor a unit. Let $A' = A/(r)$. Prove that $d - 1$ is the length of any chain of primes in A' of maximal length.

4. A *valuation ring* is an integral domain A such that for any non-zero x in the field of fractions, K, of A, at least one of x or x^{-1} lies in A. Show that (i) A is a local ring; (ii) If $A \subseteq S \subseteq K$ then S is a valuation ring; (iii) A is integrally closed.

5. (i) Let X be the set of all prime ideals of R. For each subset E of R, let $V(E)$ denote the set of all prime ideals of R which contain E. Show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology*. The topological space X is called the *prime spectrum of* R, and is written $\text{Spec}(R)$.

(ii) Draw pictures of $\text{Spec}(R)$ when $R = Z, R, C[X], R[X], Z[X]$.

(iii) For each $r \in R$, let X_r denote the complement of $V(r)$ in X. Show that they form a basis of open sets for the Zariski topology. Prove that (a) $X_r \cap X_s = X_{rs}$ (b) $X_r = \emptyset$ iff r is nilpotent (c) $X_r = X$ iff r is a unit (d) $X_r = X_s$ iff $\sqrt{(r)} = \sqrt{(s)}$ (e) X is compact (in the sense that every open covering has a finite sub-covering).

(iv) Call a topological space T *irreducible* if it is non-empty and if every pair of non-empty open sets in T intersect (equivalently every non-empty open set is dense in T). Show that Spec(R) is irreducible iff the nilradical of R is a prime ideal. [With a bit more work one can show that the irreducible components of X are the closed sets $V(P)$, where P is a minimal prime ideal of R.]

6. Give an example of a ring with only one prime ideal which is not noetherian (hence not artinian).

7. (i) Let A be a ring such that (1) for each maximal ideal M of A, the local ring A_M is noetherian; (2) for each non-zero $x \in A$, the set of maximal ideals of A which contain x is finite. Show that A is noetherian.

(ii) Give an example of a noetherian domain of infinite dimension. [This is originally due to Nagata. Let k be a field and consider $A = k[X_1, X_2, ...]$, a polynomial ring over k in a countably infinite set of indeterminates.]

8. Let A be a domain. Prove that, if A is a UFD, then every height 1 prime is principal, and that the converse holds if A is noetherian.

9. (i) Let G be a finite group of automorphisms of a ring R, and let R^G denote the subring of G-invariants, i.e. all $r \in R$ such that $\sigma(r) = r$ for all $\sigma \in G$. Show that R is integral over R^G. [Hint: if $r \in R$, observe r is the root of the polynomial $\Pi_{\sigma \in G}(X - \sigma(r))$.]

(ii) Let S be a multiplicatively closed subset of R such that $\sigma(S) \subseteq S$ for all $\sigma \in G$, and let $S^G = S \cap R^G$. Show that the action of G on R extends to an action on $S^{-1}R$, and that $(S^G)^{-1}R^G \cong (S^{-1}R)^G$.

10. Let P be a prime idea of A^G, and let Π be the set of prime ideals whose contraction is P (recall that if $f : A \to B$ is a homomorphism of rings and if I is an ideal of B, then $f^{-1}(I)$ is always an ideal of A, called the *contraction* of I). Show that G acts transitively

on Π. In particular Π is finite.

11. Let A be an integrally closed domain, K its field of fractions and L a finite normal separable extension of K. Let G be the Galois group of L over K and let B be the integral closure of A in L. Show that $\sigma(B) = B$ for all $\sigma \in G$ and that $A = B^G$.

12. For A, K as in Q.11, let L be any finite extension field of K, and let B be the integral closure of A in L. Show that if P is any prime ideal of A, then the set of prime ideals Q of B which contract to P is finite. [In topological language this says the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has finite fibers. Hint: reduce to the two cases (1) L is separable over K and (2) L is purely inseparable over K.]

13. (i) (The prime avoidance theorem) Let R be a ring and I a subset of R that is stable under addition and multiplication. Let $P_1, \ldots, P_n$ be ideals and $P_3, \ldots, P_n$ prime. Show that if I is not a subset of P_j for all j, then there is an $r \in I$ such that $r \notin P_j$ for all j; or equivalently, if $I \subset \bigcup_{i=1}^n P_i$, then $I \subset P_i$, for some i.

(ii) Suppose R is noetherian and P a prime of height at least 2. Prove that P is a union of height 1 primes, but not finitely many.

14. Let R be a noetherian ring, Prove that TFAE:

(i) R has only finitely many primes.

(ii) R has only finitely many height 1 primes.

(iii) R is semi-local of dimension 1. [A ring with only a finite number of maximal ideals is called *semi-local*.]

S.Martin@dpmms.cam.ac.uk