III **Commutative Algebra** **Michaelmas Term 2020**

EXAMPLE SHEET 2

All rings R are commutative with a 1.

1. Let S be a multiplicatively closed subset of a ring R, and let M be a finitely-generated R-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

2. Let I be an ideal of R, and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

3. A multiplicatively closed subset S of R is *saturated* when $xy \in S$ iff both x and y are in S. Prove that

(i) S is saturated iff $R \setminus S$ is a union of prime ideals.

(ii) If S is any multiplicatively closed subset of R, there is a unique smallest saturated multiplicatively closed subset S' containing S, and that S' is the complement in R of the union of the prime ideals which do not meet S. [We call S' the *saturation* of S.]

If $S = 1 + I$ for some ideal I, find S'.

4. Let S, T be two multiplicatively closed subsets of R, and let U be the image of T in $S^{-1}R$. Show that the rings $(ST)^{-1}R$ and $U^{-1})S^{-1}R)$ are isomorphic.

5. Let R be a ring. Suppose that for each prime ideal P the local ring R_P has no non-zero nilpotent element. Show that R has no non-zero nilpotent element. If each R_P is an integral domain, is R necessarily an integral domain?

6. Let S, T be multiplicatively closed subsets of R, such that S is contained in T. Let $\phi : S^{-1}R \to T^{-1}R$ be the homomorphism which maps each $a/s \in S^{-1}R$ ro a/s considered as an element of $T^{-1}R$. Show that the following statements are equivalent:

(i) ϕ is bijective.

(ii) For each $t \in T$, $t/1$ is a unit in $S^{-1}R$.

(iii) For each $t \in T$ there exists $x \in R$ such that $xt \in S$.

(iv) T is contained in the saturation of S (see Q.3).

(v) Every prime ideal which meets T also meets S.

7. Suppose $R \neq 0$ and let Σ be the set of all multiplicatively closed subsets S of R such that 0 does not belong to S. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal iff $R \setminus S$ is a minimal prime ideal of R.

The set S_0 of all non-zero-divisors in R is a saturated multiplicatively closed subset of R. Hence the set D of zero-divisors in R is a union of prime ideals.. Show that every minimal prime ideal of R is contained in D.

The ring $S_0^{-1}R$ is called the *total ring of fractions* of R. Prove that

(i) S_0 is the largest multiplicatively closed subset of R for which the homomorphism $R \to S_0^{-1}R$ is injective.

(ii) Every element in $S_0^{-1}R$ is either a zero-divisor or a unit.

(iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions (i.e. $R \to S_0^{-1}R$ is bijective.)

8. Show that $(Z/mZ) \otimes_Z (Z/nZ) = 0$ if m, n are coprime.

9. Let M be a finitely-generated R-module and let F be the (free) module R^n. Suppose $\varphi : M \to F$ is a surjective homomorphism. Show that ker φ is finitely-generated. Show that every set of generators of F is a basis of F. Deduce that every set of n generators of F is a basis of F. Deduce that every set of generators of F has at least n elements.

10. An R-module N is called *flat* if $N \otimes -$ retains exactness for all short exact sequences. If M, N are flat R-modules , show that $M \otimes_R N$ is also flat. If A is a flat R-algebra and N is a flat A-module, show that N is flat as an R-module.

11. Let M_i $(i \in I)$ be a family of R-modules, and let M be their direct sum. Show that M is flat iff each M_i is flat. Deduce that, if $R[X]$ is the ring of polynomials in one indeterminate over the ring R, then $R[X]$ is a flat R-algebra.

12. Let M be an R-module. Show that the $S^{-1}R$-modules $S^{-1}M$ and $S^{-1}R \otimes_R M$ are isomorphic; more precisely there exists a unique isomorphism $f : S^{-1}R \otimes M \to S^{-1}M$ such that $f((r/s) \otimes m) = rm/s$ for all $r \in R, m \in M, s \in S$. We have already seen in (2.3)

that $S^{-1}-$ is exact: deduce that $S^{-1}R$ is a flat R-module.

13. Let A be a local ring, and M, N f.g. R-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$. [Hint: if P is the maximal and $k = R/P$ the residue field, show $k \otimes_R M \cong M/PM$. Then apply Nakayama's lemma.]

14. Let A be an integral domain and M an A-module. Call an element $x \in M$ a *torsion element* of M if its annihilator is non-zero (i.e. if x is killed by some non-zero element of A.). Show that the torsion elements of M form a submodule of M (called the *torsion submodule* of M). Denote this by $T(M)$. If $T(M) = 0$, the module is said to be *torsion-free*. Show that

(i) If M is any A-module, then $M/T(M)$ is torsion-free.

(ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

(iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T(M') \to T(M) \to T(M'')$ is exact.

15. Let S be a multiplicatively closed subset of an I.D. A . Show that $T(S^{-1}M) = S^{-1}(T(M))$. Deduce that TFAE:

(a) M is torsion-free.

(b) M_P is torsion free for all prime ideals P.

(c) M_P is torsion-free for all maximal ideals P.

16. (a) Let S be a multiplicatively closed subset of R and $f : R \to S^{-1}R$ the natural homomorphism ($r \mapsto r/1$). Let $\mathcal{C}$ be the set of contracted ideals in R and $\mathcal{E}$ the set of extended ideals in $S^{-1}R$ (notation and basic properties are in [AM, 1.17, 1.18]). If I is an ideal of R, then the extension I^e in $S^{-1}R$ is $S^{-1}I$. If J is another ideal of R, define their *ideal quotient* $(I : J)$ to be $\{r \in R : rJ \subseteq I\}$. (Hence $(0 : J)$ is just the annihilator of J.)

(i) Show that every ideal in $S^{-1}R$ is an extended ideal.

(ii) Show that $I^{ec} = \bigcup(I : (s))$ (as $s \in S$), hence that $I^e = (1)$ iff I meets S.

(iii) Show $I \in \mathcal{C}$ iff no element of S is a zero-divisor in R/I.

(iv) Deduce the prime ideals of $S^{-1}R$ are in 1-1 correspondence with the prime ideals of R which don't meet S.

(v) Show that the operation S^{-1} commutes with the formation of finite sums, products, intersections and taking radicals.

(b) Let S be a multiplicatively closed subset of R and let Q be a P-primary ideal. Show that

(i) If $S \cap P \neq \emptyset$ then $S^{-1}Q = S^{-1}R$;

(ii) If $S \cap P = \emptyset$ then $S^{-1}Q$ is $S^{-1}P$-primary and its contraction in R is Q. Hence primary ideals correspond to primary ideals in the correspondence of (a)(iv) between ideals in $S^{-1}R$ and contracted ideals in R.

17. The *support* of the R-module M is defined to be the set $\{P \in \mathrm{Spec}(R) : M_P \neq 0\}$. Prove the following results:

(i) $M \neq 0$ iff $\mathrm{Supp}(M) \neq \emptyset$.

(ii) $V(I) = \mathrm{Supp}(R/I)$.

(iii) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

(iv) If $M = \sum M_i$ then $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}\,(M_i)$.

(v) If M is f.g. then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$, hence is a closed subset of $\mathrm{Spec}(R)$. If I is an ideal, then $\mathrm{Supp}(M/IM) = V(I + \mathrm{Ann}(M))$.

(vi) If M, N is f.g. then $\mathrm{Supp}\,(M \otimes_R N) = \mathrm{Supp}(M) \cap \mathrm{Supp}\,(N)$.

(vii) If $f : R \to S$ is a ring homomorphism and M is f.g. then $\mathrm{Supp}(S \otimes_R M) = {f^*}^{-1}(\mathrm{Supp}(M))$.

S.Martin@dpmms.cam.ac.uk