Part III Commutative Algebra MT 2020

EXAMPLE SHEET 1

All rings, A, are commutative with a 1.

1. Prove that the direct product of finitely many noetherian rings is noetherian.

2. A module is *artinian* if it satisfies the descending chain condition (DCC) on submodules. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Show that

(i) M is noetherian if and only if both M' and M'' are noetherian.

(ii) M is artinian if and only if both M' and M'' are artinian.

Deduce that if M_i $(1 \leq i \leq n)$ are noetherian (resp. artinian) A-modules, so is $\bigoplus_{i=1}^n M_i$.

3. Show that the set of prime ideals in a non-zero ring possesses a minimal member (with respect to inclusion).

4. By considering trailing coefficients (i.e. without appeal to Cohen's theorem), prove that if a ring A is noetherian then the ring of formal power series in the indeterminate X, $A[[X]]$, is noetherian.

(i) If $A[X]$ is noetherian, is A necessarily noetherian?

(ii) If $A[[X]]$ is noetherian, is A necessarily noetherian?

5. (Kaplansky) Let P be a prime ideal in $A[[X]]$ and let P^* be the image of P in the natural homomorphism $A[[X]] \to A$ obtained by mapping X to 0. Show that P is f.g. iff P^* is f.g. If P^* is generated by n elements, show that P can be generated by $n+1$ elements, and by n if $X \notin P$. Use this to show that if A is a principal ideal domain then $A[[X]]$ is a UFD. [Remark: the theorem one would really like is not true: it is possible for A to be a UFD while $A[[X]]$ is not.]

6. Which of the following rings are noetherian?

(i) the ring of rational functions of z having no pole on the circle $|z| = 1$.

(ii) the ring of power series in z with a positive radius of convergence.

(iii) the ring of power series in z with an infinite radius of convergence.

(iv) the ring of polynomials in z whose first k derivatives vanish at the origin (k is some fixed integer).

(v) the ring of polynomials in z, w, all of whose partial derivatives with respect to w vanish for $x = 0$.

[In all cases the coefficients are complex numbers.]

7. Let M be a noetherian A-module and θ be an endomorphism.

(i) Prove that if θ is surjective then it is an isomorphism. injective.

(ii) If M is artinian and θ is injective, show that again θ is an isomorphism.

[Hint: in (i) consider the submodules ker θ^n; in (ii), consider the quotient modules coker θ^n.]

8. Let A be a Noetherian ring and f be a power series in $A[[X]]$. Prove that f is nilpotent if and only if all its coefficients are nilpotent.

9. (i) For any A-module, let $M[X]$ denote the set of all polynomials in X with coefficients in M, i.e. expressions of the form

$$m_0 + m_1 X + \cdots + m_r X^r$$

for $m_i \in M$. Defining the product of an element of $A[X]$ with an element of $M[X]$ in the obvious way, show that $M[X]$ is an $A[X]$-module. [Once we have covered tensor products, you can easily show that $M[X] \cong A[X] \otimes_A M$.]

(ii) If P is a prime ideal in A show that $P[X]$ is a prime ideal in $A[X]$. If Q is a maximal ideal of A, is $Q[X]$ a maximal ideal of $A[X]$?

(iii) Let M be a noetherian A-module. Show that $M[X]$ is a noetherian $A[X]$-module.

10. Show that r lies in the Jacobson radical of A iff $1 - rs$ is a unit for all s in A.

11. Prove that any field K which is finitely generated as a ring is a finite field. [Hint: why must K have characteristic $p > 0$?]

12. Let I be an ideal contained in the Jacobson radical of A, and let M be an A-module

and N be a finitely generated A-module. Let θ be an A-module map from M to N. Show that if the induced map M/IM to N/IN is surjective then θ is surjective.

13. In the ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence show the set of zero-divisors in A is a union of prime ideals.

14. A *composition series* for a module M is a chain

$$M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

(strict inclusions) of submodules beginning at M and ending at 0, such that each M_i/M_{i+1} is irreducible (i.e. has no proper submodule). The *length* of the chain is the number of links, i.e. n. The *Jordan-Hölder theorem* asserts that if M has a composition series, then any chain of submodules can be refined to a composition series, and any two composition series have the same length. We then say that M has *finite length*.

(i) Prove the stated result. Show also that M has a composition series iff M satisfies both chain conditions.

If $\mathcal{C}$ is a class of A-modules, let λ be a function on $\mathcal{C}$ with coefficients in Z (more generally with values in an abelian group). We call λ *additive* if, for each s.e.s. $0 \to M' \to M \to M'' \to 0$ in which all the terms belong to $\mathcal{C}$, we have $\lambda(M') - \lambda(M) + \lambda(M'') = 0$.

(ii) Show that the length $\ell(M)$ is an additive function on the class of all A-modules of finite length.

(iii) For vector spaces over k, show that the following conditions are equivalent: (1) finite dimension (2) finite length (3) ACC (4) DCC. Moreover if these conditions are satisfied, length = dimension.

(iv) Deduce that if A is a ring in which the zero ideal is a product $P_1 \dots P_n$ of (not necessarily distinct) maximal ideals, then A is noetherian iff A is artinian.

15. Show that an artinian ring is noetherian. [Show that it has a finite number of maximal ideals and then appeal to Q.14(iv).]

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