CATEGORY THEORY EXAMPLES 1

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1. Let L be a distributive lattice (i.e. a partially ordered set with finite joins and meets — including the empty join 0 and the empty meet 1 — satisfying the distributive law

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in L$). Show that there is a category $\mathbf{Mat}_L$ whose objects are the natural numbers, and whose morphisms $n \to p$ are $p \times n$ matrices with entries from L, where we define 'multiplication' of such matrices by analogy with that of matrices over a field, interpreting $\wedge$ as multiplication and $\vee$ as addition. Show also that if L is the two-element lattice $\{0, 1\}$, then $\mathbf{Mat}_L$ is equivalent to the category $\mathbf{Rel}_f$ of finite sets and relations between them.

2. (i) Show that the assertion 'Every small category has a skeleton' implies the axiom of choice. [Given a family $(A_i \mid i \in I)$ of nonempty sets, consider a suitable category whose objects are pairs (i, a) with $a \in A_i$.]
(ii) Show that the assertion 'If $\mathcal{C}_0$ is a skeleton of a small category $\mathcal{C}$, then $\mathcal{C} \simeq \mathcal{C}_0$' implies the assertion that, given a family $(A_i \mid i \in I)$ of nonempty sets, we can find a family $(A'_i \mid i \in I)$ where each A'_i is a nonempty finite subset of A_i. [In standard ZF set theory, this is equivalent to the full axiom of choice. Take a category whose objects are the members of $I \times \{0, 1\}$, with the morphisms $(i, j) \to (i, k)$ being formal finite sums $\sum_{s=1}^{t} n_s a_s$ where all the a_s are in A_i and the n_s are integers with $\sum_{s=1}^{t} n_s = k - j$.]

3. By an *automorphism* of a category $\mathcal{C}$, we of course mean a functor $F \colon \mathcal{C} \to \mathcal{C}$ with a (2-sided) inverse. We say an automorphism F is *inner* if it is naturally isomorphic to the identity functor. [To see the justification for this name, think about the case when $\mathcal{C}$ is a group.]
(i) Show that the inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]
(ii) If 1 is a terminal object of $\mathcal{C}$ (i.e. an object such that for any A there is a unique morphism $A \to 1$), show that $F(1)$ is also a terminal object (and hence isomorphic to 1) for any automorphism F of $\mathcal{C}$. Deduce that, for any automorphism F of $\mathbf{Set}$, there is a *unique* natural isomorphism from the identity to F. [*Hint*: Yoneda!]
(iii) Let $\mathcal{C}$ be a full subcategory of $\mathbf{Top}$ containing the singleton space 1 and the *Sierpiński space* S, i.e. the two-point space $\{0, 1\}$ in which $\{1\}$ is open but $\{0\}$ is not. Show that S is, up to isomorphism, the unique object of $\mathcal{C}$ which has precisely three endomorphisms, and deduce that $FS \cong S$ for any automorphism F of $\mathcal{C}$. Show also that there is a unique natural isomorphism $\alpha \colon U \to UF$, where $U \colon \mathcal{C} \to \mathbf{Set}$ is the forgetful functor. By considering naturality squares of the form

$$\begin{array}{ccc} UX & \xrightarrow{\alpha_X} & UFX \\ \Big\downarrow{\scriptstyle Uf} & & \Big\downarrow{\scriptstyle UFf} \\ US & \xrightarrow{\alpha_S} & UFS \end{array}$$

where f is continuous, deduce that, if $\mathcal{C}$ also contains a space in which not every union of closed sets is closed, then α_S is continuous. Hence show that F is (uniquely) naturally isomorphic to the identity functor.
(iv) If $\mathcal{C}$ is the category of finite topological spaces, show that the group of inner automorphisms of $\mathcal{C}$ has index 2 in the group of all automorphisms.

4. (i) A monomorphism $f : A \to B$ in a category is said to be *strong* if, for every commutative square

$$\begin{array}{ccc} C & \xrightarrow{h} & A \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ D & \xrightarrow{k} & B \end{array}$$

with g epic, there exists a (necessarily unique) $t\colon D \to A$ such that $ft = k$ and $tg = h$. Show that every regular monomorphism is strong.
(ii) Let $\mathcal{C}$ be the finite category with four objects A, B, C, D and non-identity morphisms $f\colon A \to B$, $g : B \to C$, $h : B \to C$, $k : A \to C$, $l : D \to B$ and $m : D \to C$ satisfying $gf = hf = k$ and $gl = kl = m$. Show that the morphism f is strong monic but not regular monic.
(iii) Let $\mathbf{2}$ be the category with two objects $0, 1$ and one non-identity morphism $0 \to 1$, and let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ (where $\mathcal{C}$ is as in (ii)) on the objects A, B and C. Find an example of an epimorphism in the functor category $[\mathbf{2}, \mathcal{D}]$ which is not pointwise epic. [Later in the course we shall prove that if $\mathcal{D}$ has pushouts then epimorphisms in any functor category $[\mathcal{C}, \mathcal{D}]$ coincide with pointwise epimorphisms.]

5. Let $(f\colon A \to B, g\colon B \to C)$ be a composable pair of morphisms.
(i) If both f and g are monic (resp. strong monic, split monic), show that the composite gf is too.
(ii) If gf is monic (resp. strong monic, split monic), show that f is too.
(iii) If gf is regular monic and g is monic, show that f is regular monic.
(iv) Let $\mathcal{C}$ be the full subcategory of **AbGp** whose objects are groups having no elements of order 4 (though they may have elements of order 2). Show that multiplication by 2 is a regular monomorphism $\mathbf{Z} \to \mathbf{Z}$ in $\mathcal{C}$, but that its composite with itself is not. [*Hint*: first show that equalizers in $\mathcal{C}$ coincide with those in **AbGp**.] In the same category, find a pair of morphisms (f, g) such that gf is regular monic but f is not.

6. A morphism $e\colon A \to A$ is called *idempotent* if $ee = e$. An idempotent e is said to *split* if it can be factored as fg where gf is an identity morphism.
(i) Show that an idempotent e splits iff the pair $(e, 1_{\mathrm{dom}\ e})$ has an equalizer, iff the same pair has a coequalizer.
(ii) Let $\mathcal{E}$ be a class of idempotents in a category $\mathcal{C}$: show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f : \mathrm{dom}\ e \to \mathrm{dom}\ d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [*Hint*: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object e is not $1_{\mathrm{dom}\ e}$, in general.]
(iii) If $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \to \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.
(iv) If all idempotents in $\mathcal{C}$ split, $\mathcal{C}$ is said to be *Cauchy-complete*; the *Cauchy-completion* $\widehat{\mathcal{C}}$ of an arbitrary category $\mathcal{C}$ is defined to be $\mathcal{C}[\check{\mathcal{E}}]$, where $\mathcal{E}$ is the class of all idempotents in $\mathcal{C}$. Verify that the Cauchy-completion of a category is indeed Cauchy-complete.
(v) Show that if $\mathcal{D}$ is Cauchy-complete (for example, if $\mathcal{D}$ has equalizers), then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\widehat{\mathcal{C}}, \mathcal{D}]$ are equivalent.

7. (i) Let $\mathcal{C}$ be a small category and $F\colon \mathcal{C} \to \mathbf{Set}$ a functor. F is said to be *irreducible* if, whenever we are given a family of functors $(G_i \mid i \in I)$ and an epimorphism $\alpha : \coprod_{i \in I} G_i \to F$ (where $\coprod$ denotes disjoint union, i.e. coproduct in $[\mathcal{C}, \mathbf{Set}]$), there exists $i \in I$ such that the restriction of α to G_i is still epimorphic. Show that F is irreducible iff there is an epimorphism $\mathcal{C}(A, -) \to F$ for some $A \in \mathrm{ob}\ \mathcal{C}$. [For the purposes of this question, you may assume the result that epimorphisms in $[\mathcal{C}, \mathbf{Set}]$ coincide with pointwise epimorphisms; cf. the remark at the end of question 4.]
(ii) Deduce that F is irreducible and projective iff there is a split epimorphism $\mathcal{C}(A, -) \to F$ for some A.
(iii) Hence show that if $\mathcal{C}$ is Cauchy-complete, then the irreducible projectives in $[\mathcal{C}, \mathbf{Set}]$ are exactly the representable functors.
(iv) Deduce that if $\mathcal{C}$ and $\mathcal{D}$ are small categories then the functor categories $[\mathcal{C}, \mathbf{Set}]$ and $[\mathcal{D}, \mathbf{Set}]$ are equivalent iff their Cauchy-completions $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ are equivalent.

8. A functor $F\colon \mathcal{C} \to \mathbf{Set}$ is called a *monofunctor* if $F(f)$ is a monomorphism (that is, injective) for every morphism f of $\mathcal{C}$. Show that the following conditions on a small category $\mathcal{C}$ are equivalent:
(i) Every morphism of $\mathcal{C}$ is monic.
(ii) Every representable functor $\mathcal{C} \to \mathbf{Set}$ is a monofunctor.
(iii) Every functor $\mathcal{C} \to \mathbf{Set}$ is an epimorphic image of a monofunctor.
Under what hypotheses on $\mathcal{C}$ is every functor $\mathcal{C} \to \mathbf{Set}$ a monofunctor?