ALGEBRAIC TOPOLOGY (PART III) MICHAELMAS 2019

EXAMPLE SHEET 1

Examples Class 1: Friday November 1, 3:30-5:00 in MR 4. I will mark problems 7, 8, and 11. Hand work in by 5pm on October 30 if you would like it marked.

1. Let $\sigma_1, \sigma_2 : [0,1] \to \mathbb{R}$ be given by $\sigma_1(x) = x, \sigma_2(x) = 1 - x$. Identifying $[0,1]$ with Δ^1 gives a cycle $\sigma_1 + \sigma_2$ in $C_1(\mathbb{R})$. Find an $x \in C_2(\mathbb{R})$ with $dx = \sigma_1 + \sigma_2$.

2. Suppose (C_*, d) is a chain complex over R. We say C is *contractible* if there is an R-linear map $H : C_* \to C_{*+1}$ which satisfies $dH + Hd = \mathrm{id}_C$. If C is contractible, show that $H_*(C) = 0$. Show that $\widetilde{S}_*(\Delta^n)$ is contactible for all $n \geq 0$.

3. If X is a space, the *cone on* X is defined to be $CX := X \times I / X \times 0$. Show that CX is contractible. The *suspension of* X is defined to be $\Sigma X := CX / X \times 1$. Express $\widetilde{H}_*(\Sigma X)$ in terms of $\widetilde{H}_*(X)$.

4. If $0 \to H \to G \to \mathbb{Z}^n \to 0$ is an exact sequence of abelian groups, show that $G \simeq H \oplus \mathbb{Z}^n$. What are the possible isomorphism types of G in the exact sequences below?
$$0 \to \mathbb{Z}/4 \to G \to \mathbb{Z}/4 \to 0 \qquad\qquad 0 \to \mathbb{Z} \to G \to \mathbb{Z}/4 \to 0$$

5. If $f : (C, d) \to (C', d')$ is a chain map, the *mapping cone* of f is the chain complex $(M(f), d_f)$ whose underlying group is given by $M(f)_n = C_{n-1} \oplus C'_n$, and whose differential is given by
$$(d_f)_n = \begin{pmatrix} d_{n-1} & 0 \\ (-1)^n f_{n-1} & d'_n \end{pmatrix}.$$
Show that $(M(f), d_f)$ is a chain complex, and that if $f \sim g$, then $M(f) \sim M(g)$. Show that $H_*(M(f)) = 0$ if and only if the map $f_* : H_*(C) \to H_*(C')$ is an isomorphism.

6. Let X be the genus 2 surface shown in the figure at the end of the sheet. Compute $H_*(X, A)$ and use this to compute $H_*(X)$. What are $H_*(X, B)$ and $H_*(X, C)$?

7. Show S^{n+m+1} can be decomposed as the union of $S^n \times D^{m+1}$ and $D^{n+1} \times S^m$ along their common boundary $S^n \times S^m$. Compute $H_*(S^n \times S^m)$ and $H_*(D^{n+1} \times S^m, S^n \times S^m)$.

8. Let $V \subset S^3$ be an open subset of S^3 homeomorphic to $S^1 \times \operatorname{int} D^2$, and let $K = S^1 \times 0 \subset S^1 \times \operatorname{int} D^2 \simeq V$. Compute $H_*(S^3, K)$ and $H_*(S^3 - K)$.

9. A *vector field* on S^n is a continuous map $\mathbf{v} : S^n \to \mathbb{R}^{n+1}$ such that $\langle \mathbf{x}, \mathbf{v}(\mathbf{x}) \rangle = 0$ for all $\mathbf{x} \in S^n$. If $\mathbf{v}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in S^n$, show that id_{S^n} is homotopic to the antipodal map. Deduce that if S^n admits a nonvanishing vector field, then n is odd.

10. Suppose $f : T^2 \to T^2$ is a homeomorphism. Show that $f_* : H_1(T^2) \to H_1(T^2)$ defines an element of $GL(2, \mathbb{Z})$, and that any element of $GL(2, \mathbb{Z})$ can be realized by a homeomorphism of T^2.

11. If $f : X \to X$ is a homeomorphism, let Y be the quotient of $X \times [0, 1]$ obtained by identifying $(x, 0)$ and $(f(x), 1)$. Show there is a long exact sequence

$$\longrightarrow H_{n+1}(Y) \longrightarrow H_n(X) \xrightarrow{1-f_*} H_n(X) \longrightarrow H_n(Y) \longrightarrow$$

Compute $H_*(Y)$ when $X = S^n$ and f is the antipodal map; when $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and f is multiplication by $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$.

12. If $H_*(X)$ is a free abelian group, show that $H_*(X \times S^1) \cong H_*(X) \oplus H_{*-1}(X)$. (In fact, this is true even if $H_*(X)$ is not free.) Compute $H_*(T^n)$.

13. Given $f : (D^n, S^{n-1}) \to (D^n, S^{n-1})$, define

$$g : (D^n \times D^m, \partial(D^n \times D^m)) \to (D^n \times D^m, \partial(D^n \times D^m))$$

by $g(x, y) = (f(x), y))$. Show that $\deg g = \deg f$.

14. (*Cancellation*) Suppose (C, d) is a chain complex, that $C_n = C'_n \oplus A$, $C_{n-1} = C'_{n-1} \oplus A$, and that the component of d_n mapping A to A is the identity map. Show that (C, d) is chain homotopy equivalent to (C', d'), where $C'_i = C_i$ for $i \neq n, n-1$, $d'_i = d_i$ for $i \neq n-1, n, n+1$, and d'_{n+1} is the composition of d_{n+1} with the projection onto C'_n. (Hint: use $d^2 = 0$ to determine d'_n.)

15. Suppose C and C' are free, finitely generated chain complexes over $\mathbb{Z}$.

    (a) Use problem 14 to prove that if $H_*(C) = 0$, then C is contractible.

    (b) Suppose $f : C \to C'$ is a chain map. Using problem 5, show that f is a chain homotopy equivalence if and only $f_* : H_*(C) \to H_*(C')$ is an isomorphism.

    (c) Use problems 5 and 14 to give a proof of the Five Lemma.

J.Rasmussen@dpmms.cam.ac.uk