EXAMPLE SHEET 4

Examples Class 4: Friday January 17, 1:30-3:30 in MR 9. I will mark problems 1, 4, 8 and 10. Leave work in my box by 5pm on January 15 if you would like it marked.

1. (a) Show that there is no orientation reversing homeomorphism $f : \mathbb{CP}^2 \to \mathbb{CP}^2$.
   (b) If $f : S^2 \times S^2 \to \mathbb{CP}^2$, show that $f_* : H_4(S^2 \times S^2) \to H_4(\mathbb{CP}^2)$ has even degree.
2. Suppose M_1 and M_2 are closed connected oriented n-manifolds and that $x_i \in M_i$. Choose charts $\phi_i : U_i \to \mathbb{R}^n$, where $x_i \in U_i \subset M_i$ and $\phi_i(x_i) = 0$ with the property that $\phi_{1*}([M_1]|_{x_1}) = -\phi_{2*}([M_2]|_{x_2})$. Let $M_i' = M_i - \phi_i^{-1}(B_1(0))$, where $B_1(0)$ is the open ball of radius 1. We define the *connected sum* $M_1\#M_2 := M_1' \amalg M_2'/\sim$ where $\phi_1^{-1}(x) \sim \phi_2^{-1}(x)$ for $x \in S^{n-1}$.
   (a) Explain why $M_1\#M_2$ is a manifold.
   (b) Show that $H^i(M_1\#M_2) = H^i(M_1) \oplus H^i(M_2)$ for $0 < i < n$.
   (c) Show that there is an orientation $[M_1\#M_2]$ on M_1 such that
   $$[M_1\#M_2]|_{M_i'} = [M_i]|_{M_i'}$$
   for $i = 1, 2$.
   (d) Show that the cup product pairing on $H_i(M_1\#M_2)$ is given by
   $$((a_1, a_2), (b_1, b_2))_\# = (a_1, b_1)_1 + (a_2, b_2)_2,$$
   for $0 < |a_1| = |a_2| < n$ and $0 < |b_1| = |b_2| < n$, where $(\cdot, \cdot)_i$ is the cup product pairing on $H_*(M_i)$.
3. Suppose M is a $\mathbb{Z}$-orientable 4-manifold. If $H_1(M)$ is free over $\mathbb{Z}$, show that $H_*(M)$ and $H^*(M)$ are free over $\mathbb{Z}$. Assume that this is the case, and choose a basis $\langle a_1, \ldots, a_n\rangle$ for $H^2(M)$. Let A be the matrix whose ijth entry is (a_i, a_j). Show that A is symmetric, and that $\det A = \pm 1$.
4. Orient $\mathbb{CP}^2$ (over $\mathbb{Z}$) so that $(a, a) = 1$, where $\langle a\rangle = H^2(\mathbb{CP}^2)$, and let $\overline{\mathbb{CP}}^2$ denote the same manifold $\mathbb{CP}^2$ with the opposite orientation. Define $X_1 = \mathbb{CP}^2\#\mathbb{CP}^2$, $X_2 = \mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, and $X_3 = S^2 \times S^2$. Show that $H_*(X_1) \simeq H_*(X_2) \simeq H_*(X_3)$, but that no two of the X_i are homotopy equivalent.
5. If M is a $\mathbb{Z}$-orientable manifold of dimension $4n + 2$, show that the dimension of $H_{2n+1}(M; \mathbb{Q})$ is even.

6. (a) Suppose u_i generates $H^{n_i}(\mathbb{R}^{n_i}|0)$ $(i = 1, 2)$. If $p_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ is the projection, show that $p_1^*(u_1) \cup p_2^*(u_2)$ generates $H^{n_1+n_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}|0)$.
   (b) Now suppose that $\pi : E_i \to B$ $(i = 1, 2)$ are vector bundles over B, and that U_i is a Thom class for E_i. If $p_i : E_1 \oplus E_2 \to E_i$ is the projection, show that $p_1^*(U_1) \cup p_2^*(U_2)$ is a Thom class for $E_1 \oplus E_2$.
   (c) Deduce that $E_1 \oplus E_2$ can be oriented so that $e(E_1 \oplus E_2) = e(E_1) \cup e(E_2)$.
7. If $p \in \mathbb{C}[z_0, z_1, z_2]$ is a homogenous polynomial, we define

$$V_p = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \,|\, p(z_0, z_1, z_2) = 0\}.$$

   If p and q are chosen such that V_p and V_q are embedded submanifolds of $\mathbb{CP}^2$ which intersect transversely, show that V_p and V_q intersect in precisely $(\deg p)(\deg q)$ points.
8. Let E be the tangent bundle of $\mathbb{CP}^n$. Compute $H_*(S(E))$.
9. Construct a 3-dimensional real vector bundle over S^4 which has no nonvanishing section.
10. If M is a closed odd-dimensional manifold, show that $\chi(M) = 0$. Next, suppose M is a compact odd-dimensional manifold with boundary. Show that $\chi(M) = \frac{1}{2}\chi(\partial M)$. Conclude that $\mathbb{RP}^2$ does not bound a compact 3-manifold and that $\mathbb{CP}^2$ does not bound a compact 5-manifold. Does $\mathbb{RP}^3$ bound a compact 4-manifold? Does $\mathbb{CP}^3$ bound a compact 7-manifold?
11. Suppose $\mathbb{F}$ is a field, and M is a closed $\mathbb{F}$-oriented manifold. Let $\langle a_i \rangle$ be a basis of $H_i(M; \mathbb{F})$ and let $\langle b_i \rangle$ be the dual basis with respect to the cup product pairing. Given $f : M \to M$, let $\Lambda_f = \{(p, f(p)) \in M \times M\}$ be the graph of f.
    (a) Express $PD_{M\times M}(\Lambda_f)$ in terms of the a_i and b_j.
    (b) Define the *Lefshetz number* of f by $L(f) = \sum_{j=0}^{n}(-1)^j \operatorname{Tr} f_j^*$, where $f_j^* : H^j(M; \mathbb{F}) \to H^j(M; \mathbb{F})$. Show that $L(f) = \pm\Lambda_f \cdot \Delta$, where $\Delta \subset M \times M$ is the diagonal.
    (c) Deduce that if $L(f) \neq 0$, f must have a fixed point.
    (d) Show that any map $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ has a fixed point.

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