ALGEBRAIC TOPOLOGY (PART III) **MICHAELMAS 2019**

EXAMPLE SHEET 3

Examples Class 3: Friday November 29, 1:30-3:30 in MR 9. I will mark problems 1, 4, 6 and 9. Hand work in by 5pm on November 27 if you would like it marked.

1. Let $X = L^3_4 \times \mathbb{RP}^3$.
   (a) Write out $C^{cell}_*(X)$ and use it to compute $H_*(X)$.
   (b) Compute $H_*(X)$ using the Kunneth formula and verify it agrees with your answer in a).
   (c) Use the universal coefficient theorem and your answer to part b) to compute $H^*(X)$ and $H_*(X;\mathbb{Z}/2)$.
   (d) Use the universal coefficient theorem to compute $H_*(L^3_4;\mathbb{Z}/2)$ and $H_*(\mathbb{RP}^2;\mathbb{Z}/2)$. Use your answer to compute $H_*(X;\mathbb{Z}/2)$ and verify that it agrees with your answer in c).
2. If X is a finite cell complex, define the *Euler characteristic* $\chi(X) := \sum_k (-1)^k n_k$, where n_k is the number of k-cells in X. Prove the following properties of χ:
   (a) $\chi(X) = \mathcal{P}_{\mathbb{F}}(X)|_{t=-1}$, where $\mathbb{F}$ is any field.
   (b) $\chi(X \times Y) = \chi(X)\chi(Y)$.
   (c) If A and B are subcomplexes of X, then $\chi(A\cup B) = \chi(A) + \chi(B) - \chi(A\cap B)$.
3. Let $\Phi_{(X,A)} : H^*(X,A) \otimes H^*(Y) \to H^*(X\times Y, A\times Y)$ be given by $\Phi_{(X,A)}(a\otimes b) = a\times b$, and define $\Phi_A : H^*(A) \to H^*(A\times Y)$ similarly. Show that $\delta\Phi_{(X,A)} = \Phi_A\delta$, where δ denotes the boundary map in the appropriate LES of a pair.
4. If X is a space, let $\Delta_X : X \to X\times X$ be the *diagonal map* given by $\Delta(x) = (x,x)$. Compute $\Delta^*_{S^2} : H^*(S^2\times S^2) \to H^*(S^2)$ and $\Delta^*_{T^2} : H^*(T^2\times T^2) \to H^*(T^2)$.
5. Let $U, V \subset X$ be open sets. If $x \in H^*(X,U)$ and $y \in H^*(X,V)$, show that $x\cup y \in H^*(X, U\cup V)$. Using this, show that if X has a covering by n contractible open subsets, then $a_1\cup a_2\cup\ldots\cup a_n = 0$ whenever $a_1,\ldots a_n \in H^*(X)$ have grading > 0. Deduce that if T^k can be covered by n contractible open subsets, then $n > k$. Find a covering of T^2 by three contractible open subsets.
6. Let Σ_g be the surface of genus g, and suppose $\varphi : \Sigma_g \to \Sigma_h$, where $g < h$. Show that the map $\varphi_* : H_2(\Sigma_g) \to H_2(\Sigma_h)$ is the zero map. (Hint: consider $\varphi^* : H^1(\Sigma_h) \to H^1(\Sigma_g)$.)
7. Given $\phi : S^{2n-1} \to S^n$, let $X_\phi = S^n \cup_\phi D^{2n}$. $H^*(X_\phi) = \langle 1, x_n, x_{2n}\rangle$, where $x_i \in H^i(X_\phi)$. Thus $x_n^2 = H(\phi)x_{2n}$ for some $H(\phi) \in \mathbb{Z}$.

(a) Show that the map $[\phi] \to H(\phi)$ defines a homomorphism $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$. H is known as the *Hopf invariant.*

(b) By considering a cell decomposition of $S^n \times S^n$, show that H is nontrivial for every even n. Deduce that $\pi_{4m-1}(S^{2m})$ is infinite for all $m > 0$.

8. Let M be the Mobius bundle over S^1. Show that $M \oplus M$ is a trivial bundle. Is $M \times M$ (a bundle over $S^1 \times S^1$) trivial?

9. Let $E = TS^2$ be the tangent bundle of S^2. Show that the unit sphere bundle $S(E)$ is homeomorphic to $SO(3)$, which is also homeomorphic to $\mathbb{RP}^3$. Deduce that $e(E) = \pm 2a$, where $H^2(S^2) = \langle a \rangle$.

10. Let $E \to B$ be a real vector bundle equipped with a Riemannian metric, and let $F \subset E$ be a subbundle. Show that $F^\perp$ is a vector bundle, and that $F \oplus F^\perp \cong E$.

11. Given $\gamma : S^{n-1} \to O(k)$, let $E_\gamma = D^n_N \times \mathbb{R}^k \coprod D^n_S \times \mathbb{R}^k / \sim$, where $\sim$ identifies $(x, v) \in S^{n-1}_N \times \mathbb{R}^2$ with $(x, \gamma(x) \cdot v) \in S^{n-1}_S \times \mathbb{R}^k$. Show that E_γ is an vector bundle over S^n and that any vector bundle over S^n is isomorphic to some E_γ. Show further that $E_\gamma \simeq E_{\gamma'}$ if and only if $\gamma \sim \gamma'$.

12. Show that up to isomorphism there is a unique nontrivial k-dimensional vector bundle E_k over S^1. Use ES 1 #11 to compute $H^*(S(E_k))$.

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