Algebraic Topology Part III, 2015-16: Sheet 4

1. (a) Which of the following are orientable? (i) $\mathbb{RP}^3$ (ii) $\mathbb{RP}^2 \times \mathbb{CP}^2$ (iii) $K\#T^2$, where K is the Klein bottle ($\#$ denotes connect sum).

   (b) Prove that any manifold has an orientable double cover.

2. If $\{C_*^a, \rho_{ab}\}_{a\in A}$ is a direct system of chain complexes (C_*^a, d^a) indexed by a poset A, show that $H_k(\varinjlim(C_*^a)) = \varinjlim H_k(C_*^a)$. Deduce that the direct limit of exact sequences is exact.

3. (i) Let M be a closed connected oriented n-manifold. Show that there is a degree one map $M \to S^n$.

   (ii) If M and N are closed connected oriented manifolds of the same dimension and $f : M \to N$ has non-zero degree, is $f^* : H^*(N;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ necessarily injective?

   (iii) Prove that if a finite group G acts freely on S^n then some G-orbit is not contained in any open hemisphere. [*Hint: Construct a map* $S^n/G \to S^n$.]

4. Show that the only non-trivial cup-products in $(S^2 \times S^8)\#(S^4 \times S^6)$ are those forced by Poincaré duality. Give a space in which that conclusion would not be true.

5. (i) Show there is no map $\mathbb{CP}^2 \to \mathbb{CP}^2$ of degree -1.

   (ii) Show there is no map $\mathbb{CP}^2 \times \mathbb{CP}^2 \to \mathbb{CP}^2 \times \mathbb{CP}^2$ of degree -1.

   (iii) Let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be a map of degree 8. What can you say about n?

6. (a) Suppose $Y \subset X$ is a smooth closed submanifold of a smooth closed manifold. Using the tubular neighbourhood theorem, prove $H^*_{ct}(X\backslash Y) \cong H^*(X, Y)$.

   (b) Suppose $X \subset S^n$ is a closed codimension one smooth submanifold. Show that the complement $S^n\backslash X$ has $1 + b_{n-1}(X)$ connected components. [You may assume that the complement $S^n\backslash X$ has "finite type".]

7. (i) Show by induction on the dimension that a non-degenerate skew-symmetric bilinear form over $\mathbb{R}$ is equivalent to a direct sum of copies of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence show that any oriented closed six-manifold has even third Betti number.

   (ii) Let V be a vector space carrying a non-degenerate skew form as above. If $W \subset V$ is *isotropic*, meaning $\langle \cdot, \cdot \rangle|_{W\times W} \equiv 0$, show that $\dim(W) \leq \frac{\dim(V)}{2}$. What does this say about the cohomology classes defined by a collection of pairwise disjoint 3-dimensional submanifolds of a closed oriented six-manifold?

8. (a) Describe the long exact sequence associated to the short exact sequence

$$0 \to C_*(X;\mathbb{Z}) \xrightarrow{n} C_*(X;\mathbb{Z}) \longrightarrow C_*(X;\mathbb{Z}_n) \to 0$$

   where the first map is multiplication by $n \in \mathbb{Z}_{>0}$ and $C_i(X;G)$ denotes singular chains $\left\{\sum a_g\sigma_g \,|\, a_g \in G,\, \sigma_g : \Delta^i \to X\right\}$ with coefficients in the abelian group G. Give the corresponding cohomological sequence.

(b) Suppose $H^k(X;\mathbb{Z})$ is finitely generated and free for every k, and let $\{\xi_j\}$ be a basis for H^k. Show that the images $\{\tilde{\xi_j}\}$ of $\{\xi_j\}$ in $H^k(X;\mathbb{Z}/p)$ under the natural map (induced by $\mathbb{Z} \to \mathbb{Z}_p$) also form a basis for $H^k(X;\mathbb{Z}/p)$. Is the freeness assumption on the integral cohomology necessary?

(c) Now suppose X is a closed oriented manifold and set $a_{ij} = \int_X \xi_i \xi_j \in \mathbb{Z}$. Show that the matrix (a_{ij}) has determinant ± 1, and deduce $H^k(X;\mathbb{Z}) \cong Hom(H^{n-k}(X;\mathbb{Z}),\mathbb{Z})$.

9. (a) Let M be a smooth oriented closed manifold. Suppose the circle S^1 acts smoothly on M with discrete (hence isolated) fixed point set. Show that the number of fixed points is the Euler characteristic $\chi(M)$ of M.

   (b) Prove that if E and F are oriented vector bundles over a space X, their Euler classes satisfy $e_{E\oplus F} = e_E \cdot e_F$. Deduce that if n is even, the tangent bundle TS^n contains no non-trivial subbundle.

10. Let $n > 1$. For a continuous map $\phi : S^{2n-1} \to S^n$, let Y_ϕ be the space obtained by attaching a $2n$-cell to S^n via ϕ. Compute $H^*(Y_\phi)$. Fixing $\alpha_i \in H^i(Y_\phi)$ to be generators for $i \in \{n, 2n\}$, define $h(\phi)$ by $\alpha_n^2 = h(\phi)\alpha_{2n}$.

    (i) If ϕ is homotopic to a constant, show $h(\phi) = 0$.

    (ii) Let n be even. Fix a base-point $e \in S^n$. By considering the quotient $(S^n \times S^n)/\sim$ for $\sim$ the equivalence relation $(x,e) \sim (e,x)\ \forall x$, show that there is a map $\phi : S^{2n-1} \to S^n$ with $h(\phi) = \pm 2$. Deduce that the homotopy group $\pi_{2n-1}(S^n)$ is infinite.

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