Algebraic Topology Part III, 2018-19: Sheet 3

1. Let $E \to X$ be a vector bundle with inner product $\langle \cdot, \cdot \rangle$. Let $F \subset E$ be a vector subbundle. Prove that the orthogonal complement bundle $F^{\perp}$ is locally trivial.

2. (i) Let V be a real n-dimensional vector space. Show that an orientation of V, meaning a choice of generator of $H^n(V, V \backslash \{0\}) \cong \mathbb{Z}$, is equivalent to an orientation in the sense of linear algebra, i.e. a choice of ordered basis, where bases differing by a positive determinant matrix are equivalent. Deduce that a complex vector bundle has a canonical orientation.

   (ii) If M is an oriented smooth manifold and $Y \subset M$ is a closed smooth submanifold, show an orientation of Y determines a co-orientation of Y (i.e. an orientation of the normal bundle $\nu_{Y/M}$).

   (iii) If M is an oriented smooth manifold and $Y, Z \subset M$ are closed oriented smooth submanifolds which meet transversely, show that an ordering of Y and Z defines a co-orientation of $Y \cap Z$.

3. (i) Explain how to view the open Möbius band as a real line bundle (i.e. bundle of real rank one) over the circle, and prove that bundle is non-trivial.

   (ii) Prove that a real line bundle over S^n is trivial if $n > 1$. By considering associated double covers, deduce that isomorphism classes of real line bundles over a finite cell complex X are naturally in 1-1 correspondence with elements of $H^1(X; \mathbb{Z}/2)$.

4. Prove the Gysin sequence for a bundle $E \to B$ is an exact sequence of $H^*(B)$-modules.

5. (i) Define what it means for a vector bundle to be R-orientable, for a coefficient ring R, in such a way that any vector bundle is orientable with $\mathbb{Z}/2$-coefficients.

   (ii) By considering the Gysin sequence for the tautological real line bundle, prove that as a ring $H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[w]/(w^{n+1})$ for an element w of degree 1.

   (iii) Show that any map $\mathbb{RP}^n \to \mathbb{RP}^m$ acts trivially on reduced cohomology if $n > m$. What about if $n < m$?

   (iv) Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$ although they have additively isomorphic (co)homology.

6. (i) Let $L \to \mathbb{CP}^n$ be the tautological complex line bundle. By considering the line bundle $\pi_1^* L \otimes_{\mathbb{C}} \pi_2^* L \to \mathbb{CP}^n \times \mathbb{CP}^n$, with the $\pi_i : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ being projections to the factors, prove that the Euler class of $L \otimes_{\mathbb{C}} L$ is equal to twice the Euler class of L.

   (ii) Prove that the unit circle bundle in $L \otimes_{\mathbb{C}} L$ is homeomorphic to $\mathbb{RP}^{2n+1}$. Hence, compute the cohomology of $\mathbb{RP}^{2n+1}$ from knowledge of the cohomology of $\mathbb{CP}^n$.

7. Let $p : E \to B$ be a fibre bundle over a path-connected space B with fibre $F \cong p^{-1}(b)$. Suppose that $H^*(F)$ is finitely generated and free, and that inclusion $i : F \to E$ induces a surjection on cohomology (since B is path-connected, this holds for the inclusion of

any fibre). Pick a map $\theta : H^*(F) \to H^*(E)$ splitting i^*. Assuming that B admits a finite cover of trivialising open sets for E, prove the *Leray-Hirsch theorem*: the map

$$H^*(B) \otimes H^*(F) \to H^*(E),\ x \otimes y \mapsto p^*x \cdot \theta(y)$$

is an isomorphism of $H^*(B)$-modules. [In other words, $H^*(E)$ is a free $H^*(B)$-module, generated by a collection of classes whose restrictions to the fibre generate $H^*(F)$.]

8. Let X be a compact paracompact space. To a map f from X to the infinite Grassmannian $\mathrm{Gr}_k = \mathrm{Gr}_k(\mathbb{R}^\infty) = \bigcup_n \mathrm{Gr}_k(\mathbb{R}^n)$ we associate the pullback $f^*\mathbb{E}$ of the tautological bundle. We fix an inner product on $\mathbb{R}^\infty$ throughout.

   (i) Suppose $f_0, f_1 : X \to \mathrm{Gr}_k$ are maps with image in $\mathrm{Gr}_k(\mathbb{R}^N)$ for some N. Let $U \subset \mathrm{Gr}_k(\mathbb{R}^N) \times \mathrm{Gr}_k(\mathbb{R}^N)$ be the following neighbourhood of the diagonal:

   $$U = \{(v_1, v_2) \mid v_1 \cap v_2^\perp = \{0\}\}$$

   Show that if $f_0(x)$ and $f_1(x)$ belong to U for every $x \in X$ then $f_0^*\mathbb{E} \cong f_1^*\mathbb{E}$.

   (ii) By splitting the homotopy into many small intervals, deduce that if $f_0 \simeq f_1 : X \to \mathrm{Gr}_k$ are homotopic then $f_0^*\mathbb{E}$ and $f_1^*\mathbb{E}$ are isomorphic.

   (iii) Let $incl_j : v_j \hookrightarrow \mathbb{R}^N$ be the inclusion of k-dimensional subspaces v_j, for $j = 0, 1$, and let $\alpha : v_0 \to v_1$ be a linear isomorphism. Show that

   $$\{\gamma(t) = t \cdot (incl_0) + (1 - t) \cdot (incl_1 \circ \alpha)\}$$

   is a path of subspaces from $v_0 \oplus \{0\}$ to $\{0\} \oplus v_1$ in $\mathrm{Gr}_k(\mathbb{R}^{2N})$.

   (iv) Let $f_0, f_1 : X \to \mathrm{Gr}_k$ have image in $\mathrm{Gr}_k(\mathbb{R}^N)$ and $f_0^*\mathbb{E} \cong f_1^*\mathbb{E}$. Let $T : \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N$ be the map $(\xi, \eta) \mapsto (-\eta, \xi)$. Show that f_0 and $T \circ f_1$ are homotopic as maps from X to $\mathrm{Gr}_k(\mathbb{R}^{2N})$, and deduce that $f_0 \simeq f_1 : X \to \mathrm{Gr}_k$.

   Conclude that the set of isomorphism classes $\mathrm{Vect}_k(X)$ of rank k real vector bundles over X is in bijection with the set of homotopy classes $[X, \mathrm{Gr}_k]$.

9. Assume that the map $A \mapsto A^k$ on the unitary group $U(n)$ has degree k^n. Let G be a finitely presented group which has a non-trivial finite-dimensional unitary representation. Add one generator and one relation to G to obtain a new group G'. Show that G' also has a non-trivial finite-dimensional unitary representation. [*Hint: view the relation as a function on the unitary group and think about degrees of maps.*]

10.* (Optional) Show that the map $A \mapsto A^k$ on $U(n)$ has degree k^n. [*This is quite hard. One possibility is to first compute the degree on the torus T of diagonal matrices, and then think about a covering map $U(n)/T \times T \to U(n)$. Alternatively, prove that all preimages of a sufficiently generic diagonal matrix are diagonal, and argue from there.*]

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