Algebraic Topology Part III, 2018–19: Sheet 2

1. Is there a four-dimensional cell complex whose homology groups, written from left to right, so going from degree 0 to degree 4, are $\mathbb{Z}, 0, \mathbb{Z}, 0, \mathbb{Z}/2$?

2. (i) Let X be a cell complex and $A \subset X$ a subcomplex. Show (X, A) form a good pair.

   (ii) Let X be a cell complex and $K \subset X$ a compact subspace. Prove that K meets only finitely many (open) cells in X. Deduce that any element in the image of $H_i(K) \to H_i(X)$ lies in the image of $H_i(X_k) \to H_i(X)$ for $k \gg 0$.

3. Let $X = S^n \cup_\phi D^{n+1}$ be given by gluing an $(n+1)$-cell to an n-sphere by a map of degree $m > 1$. Show that the natural map $X \to X/S^n \cong S^{n+1}$ is trivial on homology $H_{*>0}$, but is non-trivial on cohomology $H^{*>0}$. What happens if we instead consider the inclusion map $S^n \hookrightarrow X$?

4. A *covering space* $p : E \to B$ is a map for which there is an open cover $\{V_a\}$ of B with $p^{-1}(V_a) = \amalg_b U_{a,b}$ a disjoint union and each $p|_{U_{a,b}} : U_{a,b} \to V_a$ a homeomorphism.

   (i) If p has finite fibres of cardinality d, construct a map $p^! : H_*(B) \to H_*(E)$ with $p_* \circ p^!$ multiplication by d. [You may wish to look up "lifting properties" for covering maps.]

   (ii) Considering Euler characteristics, show there is a covering map $\Sigma_g \to \Sigma_h$ if and only if $g = kh - k + 1$ for some $k \in \mathbb{N}$.

   (iii) If p is a double covering, so $d = 2$ in (i), construct a long exact sequence of homology groups *with* $\mathbb{Z}/2$*-coefficients*

$$\cdots \to H_r(B) \to H_r(E) \xrightarrow{p_*} H_r(B) \to H_{r-1}(B) \to \cdots$$

   (iv) Let $f : S^n \to S^n$ be an odd map, i.e. $f(x) = -f(-x)$. By considering an induced map on the exact sequences of (iii) associated to $S^n \to \mathbb{RP}^n$, show that f has odd degree.

5. (i) Compute the cohomology ring of the surface Σ_g for each $g \geq 0$.

   (ii) An *orientation* of Σ_g is a choice of isomorphism $H^2(\Sigma_g) \cong \mathbb{Z}$. Define the degree of a map of oriented surfaces to be the induced map on H^2. For which g is there a map $\Sigma_g \to \Sigma_1$ of positive degree? For which g is there a map $\Sigma_1 \to \Sigma_g$ of positive degree ?

6. If $f : \mathbb{C}^n \to \mathbb{C}^n$ has components the elementary symmetric functions

$$(z_1, \ldots, z_n) \mapsto (\sigma_i(\underline{z})) \qquad \sigma_1 = \sum_j z_j \qquad \sigma_2 = \sum_{i<j} z_i z_j \quad \cdots \quad \sigma_n = \prod_j z_j$$

   then prove that f extends to a map $S^{2n} \to S^{2n}$ of degree $n!$ (What happens if we replace $\mathbb{C}^n$ by $\mathbb{R}^n$?)

By considering $\mathbb{CP}^n$ as a space of homogeneous polynomials of degree n in two variables, construct a map $(\mathbb{CP}^1)^n \to \mathbb{CP}^n$ of degree $n!$ Deduce that $H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/(x^{n+1})$, for a generator $x \in H^2(\mathbb{CP}^n)$ of degree 2.

7. By considering a map to the wedge (one-point-union) of two copies of $\mathbb{CP}^2$, or otherwise, compute $H^*(\mathbb{CP}^2\#\mathbb{CP}^2)$ as a ring. Deduce that $\mathbb{CP}^2\#\mathbb{CP}^2$ is not homotopy equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$, even though they have the same (co)homology groups additively.

8. (a) Prove there is a natural surjection $H^n(X;\mathbb{Z}) \to \mathrm{Hom}(H_n(X;\mathbb{Z}),\mathbb{Z})$, which entwines the boundary maps in the LES of pairs.

   (b) Prove this natural surjection is not always an isomorphism.

   (c) Prove that $(X,A) \mapsto \mathrm{Hom}(H_n(X,A),\mathbb{Z}))$ does not satisfy the axioms of a generalised cohomology theory.

9. Show that there is a *relative cup product*

$$\smile : H^i(X,A) \times H^j(X,B) \to H^{i+j}(X, A \cup B)$$

   [*Hint: it may be helpful to consider a cochain complex $C^*_{A+B}(X)$ of cochains vanishing on simplices lying wholly in A or B, and use the proof of the small simplices theorem.*] Using this, show that if X has a cover by n contractible (i.e. homotopy equivalent to a point) open sets, then the *cup-length*

$$\max\left\{k \mid \exists\, a_1,\ldots,a_k \ \in H^{>0}(X),\ a_1 \smile \ldots \smile a_k \neq 0\right\}$$

   is strictly smaller than n. What does this say about the ring $H^*(\Sigma X)$, where Σ is the suspension operation?

10. Compute the cup-length of the torus $(S^1)^n$.

    Show there is a differentiable function on $T^2 = S^1 \times S^1$ which has fewer critical points than the sum of the Betti numbers, and laugh at Morse theorists. [Hint: define the function by drawing its level sets.] Can the same thing happen on S^2?

Ivan Smith
is200@cam.ac.uk