Part III Algebraic Geometry

Example Sheet III, 2024.

Note: If you would like to receive feedback, please turn in solutions to Questions 2, 4, and 7 by Monday, December 2nd.

1. Show that if X is a scheme and $Z \subseteq X$ is a closed irreducible subset, then there exists a unique $\eta \in Z$ such that $\overline{\{\eta\}} = Z$. [This is essentially a variant of Question 7 on Example Sheet 2, but stated as often used in lecture.]

2. Let X be a variety over $\operatorname{Spec} k$. Show:

   (a) For any closed point $p \in X$, $\dim X = \dim \mathcal{O}_{X,p}$.

   (b) If $Y \subset X$ is a closed subsets of X, then

   $$\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{X,y} \,|\, y \in Y\}.$$

   (c) If Y is a closed subset of X, then

   $$\dim Y + \operatorname{codim}(Y, X) = \dim X.$$

   (d) If $U \subseteq X$ is an open subset, then $\dim U = \dim X$.

   (e) $\dim X$ coincides with the transcendence degree of $K(X)$ over k.

   You may use the following facts from commutative algebra:

   If B is a finitely-generated k-algebra and domain and $\mathfrak{p} \subset B$ is a prime ideal, then

   $$\operatorname{height}(\mathfrak{p}) + \dim B/\mathfrak{p} = \dim B.$$

   If K is the field of fractions of B then $\dim B$ coincides with the transcendence degree of K/k.

3. For each property (a), (c) and (d) in Question 3, find a scheme X for which the property does not hold.

4. For each of the following schemes, show the claimed isomorphism. [Hint: In each case, remove some closed subset.]

   (a) $\operatorname{Cl}(\mathbb{P}^1_k \times_{\operatorname{Spec} k} \mathbb{P}^1_k) \cong \mathbb{Z}^2$.

   (b) Let $X = \operatorname{Spec} k[x, y, z]/(xy - z^2)$. Then $\operatorname{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$.

   (c) Let $X = \operatorname{Spec} k[x, y, z, w]/(xy - zw)$. Then $\operatorname{Cl}(X) = \mathbb{Z}$.

5. Let $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism. Then show that either $\varphi(\mathbb{P}^n)$ is a point or $m \geq n$ and $\dim \varphi(\mathbb{P}^n) = n$; [Hint: first show that if $m < n$, then $\varphi(\mathbb{P}^n)$ is a point. Finish by using projections from linear subspaces.]

6. Let X be a scheme, $f : \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Show that $\ker f, \operatorname{coker} f$ and $\operatorname{im} f$ are quasi-coherent. Further, if X is Noetherian and $\mathcal{F}$ and $\mathcal{G}$ are coherent, show $\ker f, \operatorname{coker} f$ and $\operatorname{im} f$ are coherent.

   Let $f : X \to Y$ be a morphism of schemes, and $\mathcal{F}$ a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_Y$-modules. Show that $f^*\mathcal{F}$ is a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules.

   Show by example that if $\mathcal{G}$ is a coherent sheaf on X, then $f_*\mathcal{G}$ need not be a coherent sheaf on Y. [Note: $f_*\mathcal{G}$ is always quasi-coherent, but this is harder to prove.]

7. Let $i : Z \to X$ be a closed immersion of schemes. Recall this means i is a homeomorphism of Z onto a closed subset of X, and the map $i^\# : \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective. We write $\mathcal{I}_{Z/X} = \ker i^\#$.

   a) Show that $\mathcal{I}_{Z/X}$ is a *sheaf of ideals* of $\mathcal{O}_X$, i.e., $\mathcal{I}_{Z/X}(U)$ is an ideal in $\mathcal{O}(U)$ for each $U \subseteq X$ open.

   b) Show that $\mathcal{I}_{Z/X}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, and is coherent if X is Noetherian.

   c) Show that there is a one-to-one correspondence between quasi-coherent sheaves of ideals of X and closed subschemes of X.

8. This is another question intended to get you to read a bit of Hartshorne which we haven't had time to cover in lecture. Try this yourself first, and if you need help, peek at II Proposition 7.3 of Hartshorne.

   Let k be an algebraically closed field, $X \subseteq \mathbb{P}^m_k$ a closed subscheme, and let $\varphi : X \to \mathbb{P}^n_k$ be a morphism (over $\operatorname{Spec} k$) induced by a line bundle $\mathcal{L}$ and sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ (so that $\mathcal{L}$ is generated by these global sections). Let $V \subseteq \Gamma(X, \mathcal{L})$ be the subspace spanned by the s_i. Then φ is a closed immersion if and only if

a) *elements of V separate points*, i.e., for any two distinct closed point $P, Q \in X$, there exists an $s \in V$ such that $s_P \in \mathfrak{m}_P \mathcal{L}_P$ but $s_Q \notin \mathfrak{m}_Q \mathcal{L}_Q$, or vice versa, and

b) *elements of V separate tangent vectors*, i.e., for each closed point $P \in X$, the set $\{s \in V \mid s_P \in \mathfrak{m}_P \mathcal{L}_P\}$ spans the k-vector space $\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P$.

[Hint: condition a) allows one to show that φ is injective as a map of sets. You may use without proof the fact that in any event X being a projective scheme implies that φ is a closed map, i.e., in particular $\varphi(X)$ is a closed subset of $\mathbb{P}^n_k$. (This follows from properties of proper morphisms, but we haven't had the chance to cover all the necessary results.) Condition b) is then used to show surjectivity of $\varphi^\# : \mathcal{O}_{\mathbb{P}^n_k} \to \varphi_* \mathcal{O}_X$.]