Part III Topics in Algebraic Geometry

Example Sheet II, 2024.

Note: If you would like to receive feedback, please turn in solutions to Questions 2, 7, and 9 by Monday, November 18th by noon in the usual way.

1. We check the details of the construction of Proj in lecture. Recall, in analogy with $\operatorname{Spec} A$, an affine scheme, we can define a *projective scheme*. Let $S = \bigoplus_{d=0}^{\infty} S_d$ be a graded ring. We will define $\operatorname{Proj} S$.

   i) We write $S^+ = \bigoplus_{d=1}^{\infty} S_d$, the *irrelevant ideal.* Define $\operatorname{Proj} S$ to be the set of all homogeneous prime ideals of S not containing the irrelevant ideal. If $I \subseteq S$ is a homogeneous ideal, let $V(I)$ denote the set of all primes in $\operatorname{Proj} S$ containing I. Show these form the closed sets of a topology on $\operatorname{Proj} S$.

   ii) We define a sheaf $\mathcal{O}$ on $\operatorname{Proj} S$. For $\mathfrak{p} \in \operatorname{Proj} S$, let $S_{(\mathfrak{p})}$ be the set of elements of the localization $S_{\mathfrak{p}}$ which are homogeneous of degree zero (i.e., a ratio of two elements of S of the same degree). Define for $U \subseteq \operatorname{Proj} S$ open the ring $\mathcal{O}(U)$ to be the set of functions

   $$s : U \to \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})}$$

   such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and every point $\mathfrak{p} \in U$ has an open neighbourhood V for which there exists $f, g \in S$ homogeneous of the same degree, $g \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, such that $s(\mathfrak{q}) = f/g$ for $\mathfrak{q} \in V$. Then show:

   a) The stalk of $\mathcal{O}$ at $\mathfrak{q}$ is $S_{(\mathfrak{q})}$.

   b) For any homogeneous $f \in S_+$, let $D_+(f)$ be the set of primes of $\operatorname{Proj} S$ not containing f. Show the sets $D_+(f)$ cover $\operatorname{Proj} S$, and for each such open set, there is an isomorphism of locally ringed spaces

   $$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}.$$

   Here $S_{(f)}$ denotes the subring of elements of degree 0 in the localization S_f. [Hint: This is a bit tricky. It is easy to define a map $\psi : D_+(f) \to \operatorname{Spec} S_{(f)}$. However, constructing its inverse θ is not so easy. Given $\mathfrak{q} \in \operatorname{Spec} S_{(f)}$, set

   $$\mathfrak{q}_n := \{s \in S_n \mid s^{\deg f}/f^n \in \mathfrak{q}\}.$$

   Show that $\mathfrak{q}_n$ is closed under addition and that $\theta(\mathfrak{q}) = \bigoplus_n \mathfrak{q}_n$ is a homogenoeous prime ideal in $D_+(f)$.]

   c) $\operatorname{Proj} S$ is a scheme.

   d) Show that if k is an algebraically closed field, then the set of closed points (i.e., points x such that the closure of $\{x\}$ is $\{x\}$) of $\operatorname{Proj} k[x_0, \ldots, x_n]$ are in one-to-one correspondence with points of $(k^{n+1} \setminus \{0\})/k^*$, with the usual action of k^* given by scalar multiplication. Show that if $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal, then the closed points of $\operatorname{Proj} k[x_0, \ldots, x_n]/I$ are in one-to-one correspondence with equivalence classes of points $(a_0, \ldots, a_n) \in (k^{n+1} \setminus \{0\})/k^*$ such that $f(a_0, \ldots, a_n) = 0$ for all $f \in I$ homogeneous.

2. Let X be a scheme, with open affine subsets $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$. Show that $U \cap V$ can be covered by open affine subschemes $\{U_i\}$ such that there are elements $f_i \in A$, $g_i \in B$ with $U_i = D(f_i) \subset U$ and $U_i = D(g_i) \subseteq V$.

   We will now define a number of properties of schemes and morphisms of schemes. This material can be found as a mixture of the text and the exercises of Chapter II, §3 of Hartshorne. Consult that text if you get stuck!

3. We say a scheme X is *irreducible* if it is irreducible as a topological space, i.e., whenever $X = X_1 \cup X_2$ with X_1, X_2 closed subsets, then either $X_1 = X$ or $X_2 = X$.

   We say a scheme X is *reduced* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

   We say a scheme X is *integral* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

   Show that a scheme is integral if and only if it is reduced and irreducible.

4. We say a scheme is *locally Noetherian* if it can be covered by affine open subsets $\operatorname{Spec} A_i$ with A_i a Noetherian ring. We say a scheme is *Noetherian* if it can be covered by a *finite* number of open affine subsets $\operatorname{Spec} A_i$ with A_i Noetherian.

   Show that a scheme X is locally Noetherian if and only if for every open affine subset $U = \operatorname{Spec} A$, A is a Noetherian ring. [Hint: This is II Prop. 3.2 in Hartshorne. Do have a go at this before you look at his proof. At least try to reduce to the following statement before you peek: given a ring A and a finite collection of elements $f_i \in A$ which generate the unit ideal, suppose A_{f_i} is Noetherian for each i. Then A is Noetherian.]

5. A morphism $f : X \to Y$ is *locally of finite type* if there exists a covering Y by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra.

   The morphism is *of finite type* if the cover of $f^{-1}(V_i)$ above can be taken to be finite.

   Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

6. For each of the properties defined above of schemes or morphisms, given an example of a scheme or morphism which violates that property. Give an example of a morphism which is locally of finite type but not of finite type.

   Remark. In the language above, we defined a variety as an integral scheme of finite type over $\operatorname{Spec} k$, for k an algebraically closed field.

7. Let X be an integral scheme. Show there is a unique point η such that the closure of $\{\eta\}$ is X; this is called the *generic point* of X. Show that the stalk of $\mathcal{O}_X$ at η is a field, called the *function field* of X, denoted by $K(X)$. Show that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is the field of fractions of A.

8. *Normalization.* A scheme is *normal* if all its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of X. Show that there is a morphism $\tilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. [A morphism $f : Z \to X$ is *dominant* if $f(Z)$ is a dense subset of X.]

9. Describe the fibres over all points of the target space for the following morphisms between affine schemes. In each case, the corresponding homomorphism of rings is the obvious one. Here k denotes a field. Which fibres are irreducible or reduced?

   1. $\operatorname{Spec} k[T, U]/(TU - 1) \to \operatorname{Spec} k[T]$.
   2. $\operatorname{Spec} k[T, U]/(T^2 - U^2) \to \operatorname{Spec} k[T]$.
   3. $\operatorname{Spec} k[T, U, V, W]/((U + T)W, (U + T)(U^3 + U^2 + UV^2 - V^2)) \to \operatorname{Spec} k[T]$.
   4. $\operatorname{Spec} \mathbb{Z}[T] \to \operatorname{Spec} \mathbb{Z}$.
   5. $\operatorname{Spec} \mathbb{Z}[T]/(T^2 + 1) \to \operatorname{Spec} \mathbb{Z}$. [Number theorists: what does the calculation you just did mean from the point of view of algebraic number theory?]
   6. $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$.

10. We say a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

   is *Cartesian* if the induced morphism $A \to B \times_D C$ is an isomorphism. Show the following diagrams are Cartesian in any category:

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_T Y & \longrightarrow & S \times_T S \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle \Gamma_f}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times_S Y & \longrightarrow & Y \times_S Y \end{array}$$

   In the first diagram one is given morphisms $X, Y \to S \to T$, and the morphism Δ is induced by the universal property of $S \times_T S$ using the identity $S \to S$ twice. In the second diagram, we assume given X, Y objects over S and a morphism $f : X \to Y$ over S. Then Δ is defined as before and Γ_f is the morphism induced by the identity $X \to X$ and $f : X \to Y$.