# Part III Algebraic Geometry

Example Sheet II, 2021.

*Note*: If you would like to receive feedback, please turn in solutions to Questions 4,5,6,11 by Thursday, November 11th. at noon. You may leave your work with my pigeon in CMS.

The first three problems are meant to make sure that the Proj construction has made sense to you. The first is about the set, the second is about the topology, and the third is about the function theory. Before attempting them, make sure that you understand how to take Proj of a polynomial ring over $\mathbb{C}$ with its standard grading.

1. Let $A$ be a $\mathbb{Z}_{\geq 0}$-graded ring; the degree 0 part will be denoted $(A)_0$. Let $f$ be a positive degree homogeneous element. Show that the localization of $A$ at $f$ is naturally $\mathbb{Z}$-graded. Construct a bijection between homogeneous primes in $A$ not containing $f$ with primes in the degree 0 part of $A_f$.

2. Let $A$ be as above and $T \subset A$ be a subset consisting of of homogeneous elements of positive degree. Recall that we defined $\mathbb{V}(T)$ as the subset of $\mathrm{Proj}(A)$ consisting of homogeneous primes containing $T$. Verify that these are the closed sets of a topology on $\mathrm{Proj}(A)$. Given an element $f$ in $A$, prove that the complement of $\mathbb{V}(f)$ is naturally homeomorphic with the Zariski spectrum of the degree 0 part of $A_f$. .

3. Let $A$ continue to be a graded ring. The homeomorphism
$$\mathbb{V}(f)^c \to \mathrm{Spec}((A_f)_0)$$
you have constructed in the previous problem endows the open sets $\mathbb{V}(f)^c$ with a sheaf of functions, namely the pullback of the structure sheaf. Let $f$ and $g$ be homogeneous positive degree elements. Describe an isomorphism between the spectrum of $(A_{fg})_0$ and a distinguished open subset in the spectrum of $(A_f)_0$, compatible with structure sheaves. These determine structure sheaves on open sets that cover $\mathrm{Proj}(A)$ and on their double overlaps. Check the cocycle condition on triple overlaps to construct $\mathrm{Proj}(A)$ as a scheme.

4. Let $A = \mathbb{C}[x_0, \dots, x_n]$ with the standard grading by degree. Identify the points of $\mathrm{Proj}(A)$ with the subset of lines through the origin in $\mathbb{C}^{n+1}$. More generally, let $a_0, \dots, a_n$ be a tuple of positive integers and let $B = \mathbb{C}[x_0, \dots, x_n]$ with the grading where $x_i$ has weight $a_i$. Establish an identification between the points of $\mathrm{Proj}(B)$ and the set of orbits in $\mathbb{C}^{n+1} \setminus \{\underline{0}\}$ under an the group action where $\lambda$ in $\mathbb{C}^\star$ acts on $(z_0, \dots, z_n)$ by sending it to $(\lambda^{a_0} z_0, \dots, \lambda^{a_n} z_n)$.

5. ($\star$) Recall that the set $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ can be described as a quotient of an open subset of $\mathbb{C}^4$ by an action of $(\mathbb{C}^\star)^2$. By mimicking the dictionary between gradings and $\mathbb{C}^\star$-actions, extend the Proj construction to rings graded by $\mathbb{Z}^2_{\geq 0}$. You must identify the appropriate notion to replace homogeneous element and homogeneous prime, define the Zariski topology, and identify a natural class of open affines endowed with a structure sheaf. Using your new bi-graded Proj construction, construct the product of projective spaces as a scheme by applying the construction to $A[x_0, x_1, y_0, y_1]$.

6. *Morphisms.* Let $k' \hookrightarrow k$ be a field extension. What is the associated map on affine schemes[1]? Let $k$ be a field and $A$ be a finitely generated $k$-algebra. Prove that specifying a morphism $\mathrm{Spec}(A) \to \mathbb{A}^n_k$ is equivalent to specifying tuple $(f_1, \dots, f_n)$ of elements in $A$. Give an example of two schemes $X$ and $Y$ and a map $X \to Y$ as *ringed spaces* that is not a morphism of schemes.

**We will now define a number of properties of schemes and morphisms of schemes. This material can be found as a mixture of the text and the exercises of Chapter II, §3 of Hartshorne. Consult that text if you get stuck!**

7. We say a scheme $X$ is *irreducible* if it is irreducible as a topological space, i.e., whenever $X = X_1 \cup X_2$ with $X_1$, $X_2$ closed subsets, then either $X_1 = X$ or $X_2 = X$.

   We say a scheme $X$ is *reduced* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ has no nilpotents.

   We say a scheme $X$ is *integral* if for every $U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain.

   Show that a scheme is integral if and only if it is reduced and irreducible.

[1] In particular, this tells you that the theory of schemes contains Galois theory as a special case.

8. We say a scheme is *locally Noetherian* if it can be covered by affine open subsets $\operatorname{Spec} A_i$ with $A_i$ a Noetherian ring. We say a scheme is *Noetherian* if it can be covered by a *finite* number of open affine subsets $\operatorname{Spec} A_i$ with $A_i$ Noetherian.

   Show that a scheme $X$ is locally Noetherian if and only if for every open affine subset $U = \operatorname{Spec} A$, $A$ is a Noetherian ring. [Hint: This is II Prop. 3.2 in Hartshorne. Do have a go at this before you look at his proof. At least try to reduce to the following statement before you peek: given a ring $A$ and a finite collection of elements $f_i \in A$ which generate the unit ideal, suppose $A_{f_i}$ is Noetherian for each $i$. Then $A$ is Noetherian.]

9. A morphism $f : X \to Y$ is *locally of finite type* if there exists a covering $Y$ by open affine subsets $V_i = \operatorname{Spec} B_i$, such that for each $i$, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$, where each $A_{ij}$ is a finitely generated $B_i$-algebra.

   The morphism is *of finite type* if the cover of $f^{-1}(V_i)$ above can be taken to be finite.

   Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of $Y$, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \operatorname{Spec} A_j$, where each $A_j$ is a finitely generated $B$-algebra.

   (Finite type is a very reasonable hypothesis to have on in practice, though objects that are only locally of finite type do occur in nature. Morphisms that are not even locally of finite type are typically pathological.)

10. *Examples.* A disconnected scheme is not irreducible. Find an example of a connected but reducible scheme. Give an example of a non-Noetherian ring whose spectrum is a Noetherian topological space. Give an example of a locally finite type morphism that is not of finite type.

11. *Normalization.* A scheme is *normal* if all its local rings are integrally closed domains. Give 3 examples of non-normal schemes.

    Let $X$ be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of $X$, let $\tilde{A}$ be the integral closure of $A$ in its quotient field, and let $\tilde{U} = \operatorname{Spec} \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of $X$. Show that there is a morphism $\tilde{X} \to X$ having the following universal property: for every normal integral scheme $Z$, and for every dominant morphism $f : Z \to X$, $f$ factors uniquely through $\tilde{X}$. [A morphism $f : Z \to X$ is *dominant* if $f(Z)$ is a dense subset of $X$.]