**III Algebra Michaelmas Term 2018**

**EXAMPLE SHEET 4**

All rings are commutative with a 1 unless stated otherwise.

1. Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$-module. Show that the following are equivalent for all $R$-modules $N$. (i) $\mathrm{Ext}^n_R(M,N)=0$, (ii) $\mathrm{Ext}^n_{R_P}(M_P,N_P)=0$ for every prime ideal $P$ of $R$, and (iii) $\mathrm{Ext}^n_{R_Q}(M_Q,N_Q)=0$ for every maximal ideal $Q$ of $R$.

2. Let $k$ be a field and let $R=k[X,Y]$. Let $M$ be the trivial $R$-module $k[X,Y]/(X,Y)$. Use the Koszul complex to calculate $\mathrm{Ext}^n_R(M,M)$ for all $n\geq 0$.

3. Let $k$ be a field. Recall that a $k$-algebra $R$, not necessarily commutative, is said to be *separable* if $R$ is projective as an $R-R$ bimodule; or equivalently, if $R$ is a bimodule direct summand of $R\otimes_k R$. Show that a separable $k$-algebra is necessarily finite dimensional as a $k$-vector space.

4. Let $K$ be a finite field extension of a field $k$. Show that it is a separable $k$-algebra exactly when it is a separable field extension of $k$.

5. Show that a ring $R$ with an exhaustive and separated filtration is an integral domain if the associated graded ring $\mathrm{gr}R$ is an integral domain. Assume that the filtration of $R$ is positive and show that $R$ is Noetherian if $\mathrm{gr}R$ is Noetherian. Is the same true for negative filtrations, for example the $P$-adic filtration of $R$ where $P$ is a prime ideal?

6. Let $R$ be a Noetherian ring with ideal $I$. Show that the Rees ring of $R$ with respect to the $I$-adic filtration is Noetherian. Let $M$ be a finitely generated $R$-module. A filtration of $M$ with respect to the $I$-adic filtration of $R$ is said to be *good* if its Rees module $Rees(M)$ is a Noetherian $Rees(R)$-module. Show that this is equivalent to it being *stable* (i.e there is some $J$ such that $M_{-J-j}=I^jM_{-J}$ for all $j>0$).

7. (Artin, Rees) Let $R$ be a Noetherian ring, and let $I$ be an ideal. Let $M$ be a finitely generated $R$-module with submodule $N$. Show that there exists $r\geq 0$ such that $N\cap I^aM=I^{a-r}(N\cap I^rM)$ for $a\geq r$.

8. (Krull) Let $R$ be a Noetherian local ring, and let $I$ be a proper ideal. Let $M$ be a finitely generated $R$-module. Then the intersection of all the submodules $MI^n$ is zero. In particular the intersection of all ideals $I^n$ is zero.

9. Let $k$ be a field and $f$ be a homogeneous polynomial of positive degree in the ring $R=k[X_1,\dots,X_n]$, graded in the usual way. Calculate the Hilbert polynomial for the

graded module $R/(f)$ and hence show that the degree of the Samuel function of $R/(f)$ with respect to the maximal ideal $(X_1, \ldots, X_n)$ is $n-1$.

10. Let $R$ be a Noetherian local ring with maximal ideal $P$. Show for non-zero-divisor $x$ that $d(R/(x)) \leq d(R) - 1$.

11. Let $R$ be a Noetherian local ring with maximal ideal $P$. Show that $\dim(R) \leq d(R)$. Furthermore suppose that $R$ is a regular local ring. Show that $\dim(R) = d(R)$ and that the associated graded ring of $R$ with respect to the $P$-adic filtration is isomorphic to a polynomial ring. Deduce that $R$ is an integral domain.

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