TECHNIQUES IN COMBINATORICS: EXAMPLES SHEET 2

Michaelmas Term 2014 W. T. G.

1. Let G be a graph with n vertices, αn^2 edges, and βn^3 triangles. Prove that G has an induced subgraph with $\lfloor n/2 \rfloor$ vertices, $(\alpha + o(1))n^2/4$ edges and $(\beta + o(1))n^3/8$ triangles.

2. Let X and Y be random variables with $\mathbb{E}X = a$ and $\mathbb{E}Y = b$. Prove that there is a non-zero probability that $X \geq aY/2b$ and $X \geq a/2$.

3. Let G be a bipartite graph with finite vertex sets X and Y and density δ. Suppose that the 4-cycle density of G is $\delta^4(1+\gamma)$ for a positive constant γ. Prove that there are subsets $A \subset X$ and $B \subset G$ such that $|\eta(A,B) - \delta\alpha\beta| \geq \theta$, where $\theta > 0$ depends on γ and δ only. [Hint: we have already proved this result when the graph is regular. Show first that if there are several vertices in X that do not have approximately the average degree, then the result is true and we can even take $B = Y$. This allows you to assume that the graph is approximately regular. Adapt the proof given in lectures so that it works with this weaker assumption. Don't worry if your proof is a bit on the ugly side.]

4. Let p be a prime and let $A \subset \mathbb{Z}_p$ be a set of density δ. Prove that A^8 contains at least $\delta^8 p^4$ "cuboids": that is, sequences of the form

$$(x, x+a, x+b, x+c, x+a+b, x+a+c, x+b+c, x+a+b+c).$$

5. Let X, Y and Z be finite sets and let $f : X \times Y \times Z \to \mathbb{C}$. Let $u : X \times Y \to \mathbb{C}$, $v : Y \times Z \to \mathbb{C}$ and $w : X \times Z \to \mathbb{C}$. By repeatedly applying the techniques of (i) squaring both sides, (ii) pulling out variables, (iii) applying Cauchy-Schwarz, and (iv) expanding out squares, prove an inequality of the form

$$|\mathbb{E}_{x,y,z} f(x,y,z)u(x,y)v(y,z)w(x,z)| \leq T(f)\|u\|_\infty \|v\|_\infty \|w\|_\infty.$$

[Hint: it makes things tidier if you assume that $\|u\|_\infty, \|v\|_\infty$ and $\|w\|_\infty$ are all at most 1 and simply aim for an upper bound that eliminates u, v and w altogether.]

6. Let X, Y and Z be finite sets and let $f : X \times Y \times Z \to \mathbb{C}$. Let $u : X \to \mathbb{C}$, $v : Y \to \mathbb{C}$ and $w : Z \to \mathbb{C}$. Prove an inequality of the form

$$\mathbb{E}_{x,y,z} f(x,y,z)u(x)v(y)w(z) \leq U(f)\|u\|_2\|v\|_2\|w\|_2.$$

7. Let X and Y be finite sets and let f_1, f_2, f_3, f_4 be four functions from $X \times Y$ to $\mathbb{C}$. Define the *generalized inner product* $[f_1, f_2, f_3, f_4]$ to be

$$\mathbb{E}_{x,x' \in X, y,y' \in Y} f_1(x,y)\overline{f_2(x,y')f_3(x',y)}f_4(x',y').$$

(i) Prove the *generalized Cauchy-Schwarz inequality*

$$\big|[f_1, f_2, f_3, f_4]\big| \leq \|f_1\|_\square\|f_2\|_\square\|f_3\|_\square\|f_4\|_\square.$$

[Hint: for each fixed x, x', the expectation over y and y' is the product of an expectation over y and an expectation over y'. Use that and the usual Cauchy-Schwarz inequality. That will not complete the proof, but it will get you well on your way.]

(ii) By considering the quantity $\|f + g\|_\square^4$, deduce that $\|.\|_\square$ is a norm.

8. Let G be a graph with n vertices and average degree αn. Does it follow that there are at least $\alpha^3 n^4$ quadruples (x, y, z, w) of vertices such that xy, yz and zw are all edges of G?

9. Let G be a graph with n vertices and adjacency matrix A. Since A is real and symmetric, it has an orthonormal basis of eigenvectors $u_1, \dots, u_n$. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Let us write $u \otimes v$ for the function that takes (x, y) to $u(x)v(y)$, and let us think of A as a function of two variables.

(i) Prove that $A = \sum_i \lambda_i u_i \otimes u_i$.

(ii) Give a combinatorial interpretation of the quantity $\sum_i \lambda_i^2$.

(iii) Give a combinatorial interpretation of the quantity $\sum_i \lambda_i^4$.

(iv) Prove that G has a non-negative eigenvector. [Hint: consider the map that takes a non-negative function f with $\|f\|_1 = 1$ to the function $Af/\|Af\|_1$. This is a continuous function from a simplex to itself, so has a fixed point.]

(v) If G has average degree d, then prove that the largest eigenvector of A is at least d. [Hint: let f be the constant function 1 and consider the quantity $\|Af\|_2^2$.]

(vi) Prove that G is quasirandom if and only if the second largest eigenvalue has small modulus. (Part of your task is to make this statement precise. The interpretation of graph-theoretic parameters in terms of eigenvalues is important for many reasons, one of which is that there are efficient algorithms for calculating eigenvalues.)

10. Generalize an argument from lectures to prove the following result. Let $X_1, \dots, X_k$ be finite sets and for each $i \neq j$ let G_{ij} be a quasirandom bipartite graph with vertex sets X_i and X_j and density α_{ij}. Let H be a graph with vertex set $\{v_1, \dots, v_k\}$ and let ϕ be a random map from $V(H)$ to $X_1 \cup \dots \cup X_k$ such that $\phi(v_i) \in X_i$ for each i. Then the probability that $\phi(v_i)$ is joined to $\phi(v_j)$ in G_{ij} for every pair ij such that $v_i v_j$ is an edge of H is approximately $\prod_{v_i v_j \in E(H)} \alpha_{ij}$.

11. Prove that for every positive integer k there exists a prime p and positive integers $x_1, \dots, x_k$ such that no $x_i - x_j$ with i, j distinct is a quadratic residue mod p.

12. Let A be a real symmetric matrix. Prove the inequality

$$|\langle Ax, x\rangle| \leq \|x\|_{\ell_2} \langle A^2 x, x\rangle^{1/2}.$$

Deduce that for every k we have

$$|\langle Ax, x\rangle| \leq \|x\|_{\ell_2}^{2(1-2^{-k})} \langle A^{2^k} x, x\rangle^{1/2^k}.$$

This can be helpful for proving that a sparse graph has the expected number of edges between any two large sets. If the graph is too sparse, then most 4-cycles will be degenerate, which makes the 4-cycle norm unsuitable. However, by raising the graph to a power, we can make it denser, which sometimes helps to get round the problem.

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