TECHNIQUES IN COMBINATORICS: EXAMPLES SHEET 1

Michaelmas Term 2014 W. T. G.

1. Let $\mathbb{Z}_n$ be the set of integers mod n, and for a subset $A \subset \mathbb{Z}_n$ and for $x \in \mathbb{Z}_n$, let $A+x$ denote the set $\{a+x : a \in A\}$. Given two subsets $A, B \subset \mathbb{Z}_n$ prove that there exists x such that $|(A+x) \cap B| \geq |A||B|/n$. Investigate the reverse inequality. (That is, try to find sets of given cardinality for which $\max_x |(A+x) \cap B|$ is as small as possible.)

2. Let G be a graph with n vertices and m edges. Prove that there is a partition of the vertices of G into two sets V, W such that the number of edges joining a vertex in V to a vertex in W is at least $m/2$.

3. (i) Let $a_1, \dots, a_n$ be real numbers and let $\epsilon_1, \dots, \epsilon_n$ be independent random elements of the set $\{-1, 1\}$. Prove that $\mathbb{E} \sum_i \epsilon_i a_i = 0$ and that $\mathbb{E}(\sum_i \epsilon_i a_i)^2 = \sum a_i^2$.

(ii) Let k be a positive integer. Prove that there exists a constant C depending only on k such that $\mathbb{E}(\sum_i \epsilon_i a_i)^{2k} \leq C(k)(\sum_i a_i^2)^k$. (This is the main step in one of the standard proofs of *Khintchine's inequality*.)

4. Prove the following statement. For every k there exists a finite set X and an antisymmetric relation R defined on X such that for any k elements $x_1, \dots, x_k$ of X there exists $x \in X$ such that xRx_i for each $i = 1, 2, \dots, k$. Investigate how large X needs to be. [A relation R is *antisymmetric* if there do not exist x, y with xRy and yRx.]

5. Let G be a graph with n vertices and minimum degree d. Prove that there is a set X consisting of at most $n(1+\log(d+1))/(d+1)$ vertices of G such that every vertex not in X is joined to at least one vertex in X. [If you cannot get precisely this bound, get the best bound you can.]

6. Let X be a finite set and let $A_1, \dots, A_r$ be subsets of X of size m. Prove that if $r < 2^{n-1}$, then there exists a red-blue colouring of the elements of X such that every A_i contains at least one red element and at least one blue element. What is the best bound you can find in the other direction? (That is, try to find $A_1, \dots, A_r$ of size m with r as small as possible such that for every red-blue colouring there is some i with all the elements of A_i the same colour.)

7. Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$. Prove that there exist signs $\epsilon_i = \pm 1$ such that $\|\sum_i \epsilon_i v_i\| \geq \sqrt{n}$.

8. Suppose you have n^2 lights arranged in an $n \times n$ grid, and for each row and each column of the grid you have a switch that will reverse all the lights in that row or column: that is, if they are on then it will switch them off and vice versa. Prove that no mattter what the initial arrangement of lights is, it is possible to flip the switches until the number of lights on minus the number of lights off is at least $(\sqrt{2/\pi} + o(1))n^{3/2}$.

9. Let $\delta > 0$. For each n let $f(n)$ be the largest number of subsets of $\{1, 2, \ldots, n\}$ of size $n/2$ that it is possible to find such that no two of them have an intersection of size more than $(1 + \delta)n/4$. Prove that $f(n)$ grows exponentially with n.

10. Prove that for every r and every k there exists a finite graph with girth at least r and chromatic number at least k. [The *girth* of a graph is the length of the shortest cycle in the graph, and the *chromatic number* is the smallest number of colours you need to colour all the vertices in such a way that no edge joins two vertices of the same colour.]

11. A collection of random variables $X_1, \ldots, X_N$ is called *pairwise independent* if X_i and X_j are independent whenever $i \neq j$. Let $a_1, \ldots, a_n$ be real numbers and let $\epsilon_1, \ldots, \epsilon_n$ be pairwise independent random elements of $\{-1, 1\}$, with each individual ϵ_i chosen uniformly. Prove that $\mathbb{E}(\sum_i \epsilon_i a_i)^2 = \sum_i a_i^2$.

12. Prove that there is a positive constant c such that the following holds. If $a_1, \ldots, a_n$ are real numbers with $\sum_i a_i^2 = 1$, and if $\epsilon_1, \ldots, \epsilon_n$ are independent random elements of $\{-1, 1\}$ chosen uniformly, then $\mathbb{P}[\sum_i \epsilon_i a_i| \leq 1] \geq c$.

13. Prove that for every set X of at least $4k^2$ distinct elements of $\mathbb{Z}_p$ (where p is prime) there exists $a \in \mathbb{Z}_p$ such that the set $\{ax : x \in X\}$ intersects every interval in $\mathbb{Z}_p$ of length at least p/k. [An *interval* is a set of the form $\{u, u + 1, \ldots, u + m\}$, where addition is mod p.]

14. Let A and B be subsets of $\mathbb{Z}_p$ of size αp and βp respectively. Prove that there exist $u, v \in \mathbb{Z}_p$ such that $\big||A \cap (uB + v)| - \alpha\beta p\big| \leq \sqrt{\alpha\beta p}$, where $uB + v$ denotes the set $\{ub + v : b \in B\}$.