TOPICS IN ANALYSIS (Lent 2025): Example Sheet 3

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1. For each positive integer n and $k \in \{1, 2, \ldots, n\}$, let the non-negative numbers $A_k^{(n)}$ and the 'nodes' $x_k^{(n)} \in [a, b]$ be given such that for each polynomial P, the error

$$\varepsilon_n(P) = \left| \int_a^b P(x)dx - \sum_{k=1}^{n} A_k^{(n)} P(x_k^{(n)}) \right|$$

in approximating a $\int_a^b P(x)dx$ by $\sum_{k=1}^n A_k^{(n)} P(x_k^{(n)})$ tends to zero as $n \to \infty$. Prove that $\varepsilon_n(f) \to 0$ for each continuous function f on $[a, b]$.

2. Let n be a positive integer. We proved in lectures that there are n distinct points $\alpha_1, \alpha_2, \ldots, \alpha_n \in [-1, 1]$ and n real numbers $A_1, A_2, \ldots, A_n$ such that the formula

$$\int_{-1}^{1} p(x)dx = \sum_{j=1}^{n} A_j p(\alpha_j)$$

is valid for every polynomial p of degree $\leq 2n - 1$. (The α_j are in fact the zeros of the nth Legendre polynomial over $[-1, 1]$). Is it possible to find such n distinct points in $[-1, 1]$ and numbers $A_1, A_2, \ldots, A_n$ such that the above formula is valid for every polynomial p of degree $\leq 2n$?

3. Let T_j be the jth Chebyshev polynomial. Suppose γ_j is a sequence of non-negative numbers with $\sum_{j=1}^\infty \gamma_j < \infty$. Prove that $\sum_{j=1}^\infty \gamma_j T_{3^j}$ defines a continuous function f on $[-1, 1]$ with the following property. For each n, there exist points $-1 \leq x_0 < x_1 < \ldots < x_{3^{n+1}} \leq 1$ such that, writing P_n for the partial sum $\sum_{j=1}^n \gamma_j T_{3^j}$,

$$f(x_k) - P_n(x_k) = (-1)^k \sum_{j=n+1}^{\infty} \gamma_j$$

holds for each $k = 0, 1, \ldots, 3^{n+1}$.

4. For each $f \in C([-1, 1])$, let $E_n(f)$ be the distance from f to the subspace $\mathcal{P}_n$ of polynomials of degree at most n. That is, $E_n(f) = \inf_{p \in \mathcal{P}_n} \sup_{x \in [-1,1]} |f(x) - p(x)|$. We know by the Weierstrass approximation theorem that $E_n(f) \to 0$ for each $f \in C([-1, 1])$. Using the result of Question 3, construct a function $f \in C([-1, 1])$ to show that the convergence $E_n(f) \to 0$ can be arbitrarily slow in the following sense. For any given decreasing sequence of non-negative numbers δ_n converging to zero, there exists $f \in C([-1, 1])$ such that $E_n(f) \geq \delta_n$ for all $n = 1, 2, \ldots$.

5. For $n, r \in \mathbb{Z}$ and $n \geq 1$, define $\Delta_{n,r} : [-1, 1] \to \mathbb{R}$ by $\Delta_{n,r}(x) = \max\{0,\ 1 - n|x - rn^{-1}|\}$. Sketch $\Delta_{n,r}$.

Now consider $f : [-1, 1] \to \mathbb{R}$. Show that $f_n(x) = \sum_{m=-n}^{n} f(m/n)\Delta_{n,m}(x)$ is a piecewise linear function with $f_n(r/n) = f(r/n)$.

Show that, if f is continuous, then $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

6. Use the result of Question 5 to prove that there exists a sequence of functions $\phi_n \in C([-1, 1])$, $n = 0, 1, 2, \ldots$, such that for every $f \in C([-1, 1])$, there exists a unique series $\sum_{n=0}^{\infty} a_n \phi_n$ which converges uniformly to f.

7. For each $n = 1, 2, \ldots$, let $f_n : [0, 1] \to \mathbb{R}$ be functions such that f_n converge uniformly to a function f. Suppose also that f is bounded. Prove that for any positive integer m the functions $g_n(t) = f_n(t)^m$ converge uniformly on $[0, 1]$ to $g(t) = f(t)^m$.

8. Let $B_r(z)$ denote the open ball about z with radius r in the complex plane and let $U = B_2(1) \setminus \overline{B_1(0)}$. Suppose that f is holomorphic in U.
(i) Prove that there exists a sequence of polynomials which converges to f uniformly on compact subsets of U.
(ii) Must there be a sequence of polynomials which converges to f uniformly on U?
(iii) If additionally we assume that f is holomorphic in some open set containing the closure of U, must there be a sequence of polynomials which converges to f uniformly on U?

9. Construct a sequence of polynomials which converges uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \mathrm{Re}(z) \geq 0\}$.

10. Let U be a bounded open subset of the complex plane $\mathbb{C}$ such that $\mathbb{C} \setminus U$ is connected. Prove that $f : U \to \mathbb{C}$ is holomorphic if and only if whenever K is a compact subset of U and $\epsilon > 0$ we can find a polynomial P such that

$$|f(z) - P(z)| < \epsilon$$

for all $z \in K$. (This gives yet another of several equivalent definitions of holomorphic functions.)

11. Does there exist a sequence of complex polynomials p_n such that $p_n(0) = 1$ for every $n = 1, 2, \ldots$ and $p_n(z) \to 0$ for each $z \in \mathbb{C} \setminus \{0\}$?

12. Let $A = \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$, and let $f : A \to C$ be continuous in A and holomorphic in the interior of A. If there exists a sequence of complex polynomials converging uniformly on A to f, prove that there exists a continuous function $g : \{z : |z| \leq 1\} \to \mathbb{C}$ such that g is holomorphic in $\{z : |z| < 1\}$ and $g(z) = f(z)$ for every $z \in A$. [Hint: if p_n are polynomials converging uniformly to f on A, apply the maximum modulus principle to $p_n - p_m$ over a suitable domain.]