TOPICS IN ANALYSIS (Lent 2025): Example Sheet 2

Comments, corrections are welcome at any time. a.g.kovalev@dpmms.cam.ac.uk.

1. Let $p(z) = z^2 - 4z + 3$ and let $\gamma : [0,1] \to \mathbb{C}$ be given by $\gamma(t) = p(2e^{2\pi it})$. Show that the closed path associated with γ does not pass through 0. Compute $w(\gamma, 0)$:

(i) non-rigorously direct from the definition by obtaining enough information about γ, (You could write the real and imaginary parts of $\gamma(t)$ in terms of $\cos t$ and $\sin t$ and find where and how γ crosses the real axis.)

(ii) by factoring, and

(iii) by the dog-walking lemma.

2. Let $g : S^1 \to S^1$ be a continuous map, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. If there is a continuous extension of g to the closed unit disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$ (i.e. if there is a continuous map $G : D \to S^1$ such that $G(z) = g(z)$ for each $z \in S^1$), prove that

(a) $g(z) = z$ for some $z \in S^1$.

(b) $g(z) = -z$ for some $z \in S^1$.

3. Does there exist a function $f : [0,1] \to \mathbb{R}$ with a discontinuity which can be approximated uniformly on $[0,1]$ by polynomials?

4. Let $f : [0,1] \to \mathbb{R}$ be a continuous function which is not a polynomial. If p_n is a sequence of polynomials converging uniformly to f on $[0,1]$, and $d_n = \deg p_n$, prove that $d_n \to \infty$.

5. Suppose $f : [-1,1] \to \mathbb{R}$ is $(n+1)$-times continuously differentiable on $[-1,1]$ and let $J_n = \{x_0, x_1, \ldots, x_n\}$ be a set of $n+1$ distinct points in $[-1,1]$. Let P_{J_n} be the interpolating polynomial of degree $\leq n$ determined by the requirement $P_{J_n}(x_j) = f(x_j)$ for each $j = 0, 1, 2, \ldots, n$. Let $\beta_{J_n}(x) = (x - x_0)(x - x_1) \ldots (x - x_n)$. Prove that for each $x \in [-1,1]$, there exists $\zeta \in (-1,1)$ such that

$$f(x) - P_{J_n}(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} \beta_{J_n}(x).$$

[Hint: If $x = x_j$ this holds trivially. If not, consider $g(y) = f(y) - P_{J_n}(y) - \lambda \beta_{J_n}(y)$ where λ is chosen so that $g(x) = 0$.]

Deduce that if f is infinitely differentiable in $[-1,1]$ and $\sup_{x \in [-1,1]} |f^{(n)}(x)| \leq M^n$ for some fixed constant M and all $n = 1, 2, \ldots$, then the interpolating polynomials P_{J_n} (for arbitrary choices of sets of interpolation points $J_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\} \subset [-1,1]$) converge uniformly to f on $[-1,1]$ as $n \to \infty$.

6. Fix $n \geq 1$ and let J be any set of n distinct points $\{x_1, \ldots, x_n\} \subset [-1,1]$. Let β_J be the polynomial defined by $\beta_J(x) = (x - x_1)(x - x_2) \ldots (x - x_n)$ and set

$$F(x_1, \ldots, x_n) = \sup_{x \in [-1,1]} |\beta_J(x)|.$$

By considering the nth Chebyshev polynomial or otherwise, prove that F is minimized when $x_k = \cos \frac{(2k-1)\pi}{2n}$, for $k = 1, 2, \ldots, n$.

7. It can be shown that the converse of the equal ripple criterion holds. That is, if $f \in C([0,1])$ and p is a polynomial of degree less than n which minimizes $\|f-q\|_\infty = \sup_{x\in[0,1]} |f(x)-q(x)|$ among all polynomials q of degree less than n, then there exist $n+1$ distinct points $0 \le a_0 \le a_1 < \ldots < a_n \le 1$ such that either $f(a_k)-p(a_k) = (-1)^k\|f-p\|_\infty$, for all $k = 0, 1, \ldots, n$ or $f(a_k)-p(a_k) = (-1)^{k+1}\|f-p\|_\infty$, for all $k = 0, 1, \ldots, n$ Assuming this, prove that for any given $f \in C([0,1])$ and each positive integer n, the minimizer of $\|f-q\|_\infty$ among all polynomials q of degree less than n is unique. (Recall that the existence of such a minimizer was proved in lectures.)

8. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, show that there exist polynomials p_n, $n = 1, 2, \ldots$, such that $p_n(x) \to f(x)$ for every $x \in \mathbb{R}$.

9. Let $B_n : C[0,1] \to C[0,1]$ be the Bernstein operator defined by

$$B_n f(x) = \sum_{k=0}^{n} f\big(\frac{k}{n}\big) \binom{n}{k} x^k(1-x)^{n-k}.$$

Show directly that $B_n f \to f$ uniformly on $[0,1]$ for the function $f(x) = x^3$.

10. Calculate the first five Chebyshev polynomials.

11. (i) Use orthogonality (the Gram–Schmidt method) to compute the Legendre polynomials p_n for $n = 0,\ 1,\ 2,\ 3$.

(ii) Explain why

$$\frac{d^m}{dx^m}(1-x)^n(1+x)^n$$

vanishes at $x = 1$ or $x = -1$ whenever $m < n$.

Suppose that

$$P_n(x) = \frac{d^n}{dx^n}(1-x^2)^n.$$

Use integration by parts to show that

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

for $m \neq n$. Conclude that the P_n are scalar multiple of the Legendre polynomials p_n.

(iii) Compute P_n for $n = 0,\ 1,\ 2,\ 3$ and check that these verify the last sentence of (ii).

12. If $f \in C[0,1]$ and $\int_0^1 f(x)x^n dx = 0$, for all $n = 0, 1, 2, \ldots$, prove that f is the zero function. If we only assume that $f \in C[0,1]$ and $\int_0^1 f(x)x^n dx = 0$, for all $n = 1, 2, \ldots$, does it still follow that f is the zero function?

$^+$**13.** The Chebyshev polynomials form an orthogonal system with respect to a certain positive weight function w. That is, $\int_{-1}^1 T_m(x)T_n(x)w(x)dx = 0$ whenever $m \neq n$. Work out what the weight function should be, and prove the orthogonality. [Hint: use an appropriate substitution for x.]