TOPICS IN ANALYSIS (Lent 2025): Example Sheet 1

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1. Let X be a non-compact subset of $\mathbb{R}^2$. Prove that there is a continuous unbounded function from X to $\mathbb{R}$.

2. Consider the metric space $(\mathbb{Q}, d)$ where $\mathbb{Q}$ is the set of rational numbers and d is the usual Euclidean metric. Let P be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Prove that P is closed and bounded in $\mathbb{Q}$, but not compact. Give an open cover of P which has no finite subcover.

3. Let K be a compact subset of $\mathbb{R}^2$ and let F be a closed subset of $\mathbb{R}^2$ such that $K \cap F = \emptyset$. Give two proofs that there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for every $x \in K$ and every $y \in F$, one proof directly from the compactness of K and the other using sequential compactness instead.

4. Find a metric d on $X = (0, 1]$ such that (X, d) is a complete metric space, and such that a subset of X is open with respect to d if and only if it is open with respect to the usual Euclidean metric.

5. Let $f : B \to B$ be a continuous function from the *open* disc $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ to itself. Must it have a fixed point? Does your answer change if f is a surjection?

6. Let (X, d) be a compact metric space and $f : X \to X$ a continuous function and suppose that $d(f(x), f(y)) \geq d(x, y)$ for every $x, y \in X$. Prove that f is a surjection.
[Hint: if not, pick $x \notin f(X)$, show that there is a ball about x that misses $f(X)$ and consider the sequence $x, f(x), f(f(x)), \ldots$]

7. (i) Let x be a point in the closed unit disc D, let K be a compact subset of D not containing x and define a map $f_x : K \to \partial D$ as follows. Given $y \in K$, take the line segment connecting x and y and extend it (in the x-to-y direction) until it first hits the boundary. Call this point $f_x(y)$. Prove that f_x is a continuous function.

(ii) Let $g : D \to D$ be a continuous function and let $x_0 \in D$ be a point such that $x_0 \neq g(x_0)$. Show that there is $r > 0$ such that $x \neq g(x)$ for each $x \in B_r(x_0)$. Let $h(x) = f_x(g(x))$, where f_x is defined as in (i). Prove that h continuous at x_0.

8. Let A be a 3×3-matrix with positive entries. Use Brouwer fixed-point theorem to prove that A has an eigenvector with positive entries. [Hint: use A to define a map from T to itself, where T is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.]

9. Use the Brouwer fixed point theorem to prove that there is a complex number z with $|z| \leq 1$ such that $z^4 - z^3 + 8z^2 + 11z + 1 = 0$.

10. Let $C[0,1]$ be the metric space consisting of all continuous functions $f : [0,1] \to \mathbb{R}$, with the distance $d(f,g)$ between two functions f and g defined to be the supremum of $|f(x) - g(x)|$ over all $x \in [0,1]$.

(i) Explain why this supremum is in fact a maximum.

(ii) Let X consist of all functions f in $C[0,1]$ that take values in $[0,1]$. Prove that X is not a sequentially compact subset of $C[0,1]$.

(iii) This shows that X is not compact. Prove the same result by exhibiting an open cover of X that has no finite subcover.

11. Suppose $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property that if $a, b \in \mathbb{R}$ and $f(a) < c < f(b)$, then $f(y) = c$ for some y between a and b.

(i) Must f be continuous? (Justify your answer.)

(ii) If additionally f has the property that $f^{-1}(a)$ is closed for every $a \in \mathbb{Q}$, show that f is continuous.

12. Let q be an integer ≥ 2 and n an integer ≥ 1. Denote by $\mathcal{Q}$ the set of all unordered q-tuples of points in $\mathbb{R}^n$. Thus $\mathcal{Q} = \big\{\{x_1, x_2, \dots x_q\} : x_1, x_2, \dots, x_q$ are not necessarily distinct points in $\mathbb{R}^n\big\}$. Define a function $\mathcal{G} : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ by

$$\mathcal{G}\big(\{x_1, x_2, \dots, x_q\}, \{y_1, y_2, \dots, y_q\}\big) = \inf\left\{ \left(\sum_{j=1}^{q} |y_j - x_{\sigma(j)}|^2 \right)^{1/2} : \sigma \text{ is a permutation of } 1, 2, \dots, q \right\}.$$

(i) Show that $\mathcal{G}$ is a metric on $\mathcal{Q}$.

(ii) Assume $n = 1$. Note that in this case, we may, for any point $x = \{x_1, x_2, \dots, x_q\} \in \mathcal{Q}$, choose the labelling so that $x_1 \leq x_2 \leq \ldots \leq x_q$. Assuming we have done this for all points in $\mathcal{Q}$, is it true that $\mathcal{G}\big(\{x_1, x_2, \dots, x_q\}, \{y_1, y_2, \dots, y_q\}\big) = \left(\sum_{j=1}^{q} (x_j - y_j)^2 \right)^{1/2}$?