TOPICS IN ANALYSIS (Lent 2020): Example Sheet 4

Comments, corrections are welcome at any time. `a.g.kovalev@dpmms.cam.ac.uk`.

1. Prove that $\sqrt{3}+\sqrt{5}$ and e^2 are irrational.

2. (i) Show that a real number α is algebraic if and only if it is a zero of a polynomial with rational coefficients.

(ii) Is it true that if all the roots of a polynomial are algebraic, then the polynomial must have rational coefficients? Give a proof or counterexample.

3. By considering

$$\sum_{n=0}^{\infty} \frac{b_n}{10^{n!}}$$

with $b_n \in \{1, 2\}$, give another proof that the set of transcendental numbers is uncountable.

4. (i) We work in $\mathbb{C}$. Show that there exists a sequence of polynomials P_n such that

$$P_n(z) \to \begin{cases} 1 & \text{if } |z| \le 1 \text{ and } \operatorname{Re} z \ge 0 \\ 0 & \text{if } |z| \le 1 \text{ and } \operatorname{Re} z < 0 \end{cases}$$

as $n \to \infty$.

[Hint: If Ω_1 and Ω_2 are disjoint open sets and $f(z) = 0$ for $z \in \Omega_1$ and $f(z) = 1$ for $z \in \Omega_2$, then f is holomorphic on $\Omega_1 \cup \Omega_2$.]

(ii) Show that there exists a sequence of polynomials Q_n such that

$$P_n(z) \to \begin{cases} 1 & \text{if } \operatorname{Re} z \ge 0 \\ 0 & \text{if } \operatorname{Re} z < 0 \end{cases}$$

as $n \to \infty$.

5. Find the continued fraction expansions for $100/37$ and $\sqrt{3}$.

6. Which rational with denominator less than 10 best approximates $71/49$ and why?

7. Let a, b be positive integers. Define

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \dots}}}}$$

with the a's and b's continuing to alternate.

(a) Show that x solves $ax^2 + abx - b = 0$.

(b) Now let $a = b = 1$. Show that, in this case, $x = (-1+\sqrt{5})/2$ and that if p_n/q_n is the nth convergent of the continued fraction above, then $p_n = F_n$, $q_n = F_{n+1}$, where $F_0, F_1, F_2, \ldots$ are the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

(c) Show that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}.$$

$^+$(d) Prove the identities $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $F_{2n} = F_n(F_{n-1} + F_{n+1})$. Denote by x_n the ratio F_{n+1}/F_n. Use the above identities to express x_{2n} as a rational function of x_n. Let $y_k = x_{2^k}$. Explain why the sequence (y_k) converges very rapidly to the golden ratio $(1+\sqrt{5})/2$.

8. Observe that the continued fraction obtained in the lectures for $\tan x$, $x \in [-1,1]$, remains valid in the more general case when $x = z \in \mathbb{C}$ and $|z| \leq 1$.

Hence obtain an expansion for $\tanh y$ with y real. Deduce that

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ldots}}}}.$$

Why does this give another proof that e is irrational?

9. Suppose that $f : [1,\infty) \to \mathbb{R}$ is a continuous function such that $f(nx) \to 0$ as $n \to \infty$ for each fixed x. Show that $f(t) \to 0$ as $t \to \infty$. [Hint: For $\varepsilon > 0$, consider the sets $Q_k = \{x \in [1,\infty) : |f(nx)| \leq \varepsilon\ \forall n \geq k\}$.]

10. Let A_j be a sequence of subsets of $[0,1]$ such that for each $N \geq 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0,1]$. Prove that the set S of points $x \in [0,1]$ such that $x \in A_j$ for infinitely many j is dense. Must S be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$?

11. If G is an open dense subset of $\mathbb{R}$, and $\mathbb{Q}$ is the set of rationals, show that $G \setminus \mathbb{Q}$ must be dense in $\mathbb{R}$. If we only assume G is uncountable and dense in $\mathbb{R}$, does it still follow that $G \setminus \mathbb{Q}$ is dense in $\mathbb{R}$?

12. Show that there is a dense set of real numbers x with the property that for each positive integer n, there exist integers p and q with $q \geq 2$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$