

TOPICS IN ANALYSIS (Lent 2020): Example Sheet 1

Comments, corrections are welcome at any time.

a.g.kovalev@dpmms.cam.ac.uk

1. Let X be a non-compact subset of \mathbb{R}^2 . Prove that there is a continuous unbounded function from X to \mathbb{R} .
2. Consider the metric space (\mathbb{Q}, d) where \mathbb{Q} is the set of rational numbers and d is the usual Euclidean metric. Let P be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Prove that P is closed and bounded in \mathbb{Q} , but not compact. Give an open cover of P which has no finite subcover.
3. Let K be a compact subset of \mathbb{R}^2 and let F be a closed subset of \mathbb{R}^2 such that $K \cap F = \emptyset$. Give two proofs that there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for every $x \in K$ and every $y \in F$, one proof directly from the compactness of K and the other using sequential compactness instead.
4. Find a metric d on $X = (0, 1]$ such that (X, d) is a complete metric space, and such that a subset of X is open with respect to d if and only if it is open with respect to the usual Euclidean metric.
5. Let $f : B \rightarrow B$ be a continuous function from the *open* disc $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ to itself. Must it have a fixed point? Does your answer change if f is a surjection?
6. Let (X, d) be a compact metric space and $f : X \rightarrow X$ a continuous function and suppose that $d(f(x), f(y)) \geq d(x, y)$ for every $x, y \in X$. Prove that f is a surjection. [Hint: if not, pick $x \notin f(X)$, show that there is a ball about x that misses $f(X)$ and consider the sequence $x, f(x), f(f(x)), \dots$]
7. (i) Let x be a point in the closed unit disc D , let K be a compact subset of D not containing x and define a map $f_x : K \rightarrow \partial D$ as follows. Given $y \in K$, take the line segment connecting x and y and extend it (in the x -to- y direction) until it first hits the boundary. Call this point $f_x(y)$. Prove that f_x is a continuous function.
(ii) Let $g : D \rightarrow D$ be a continuous function and let $x_0 \in D$ be a point such that $x_0 \neq g(x_0)$. Show that there is $r > 0$ such that $x \neq g(x)$ for each $x \in B_r(x_0)$. Let $h(x) = f_x(g(x))$, where f_x is defined as in (i). Prove that h is continuous at x_0 .
8. Let A be a 3×3 -matrix with positive entries. Use Brouwer fixed-point theorem to prove that A has an eigenvector with positive entries. [Hint: use A to define a map from T to itself, where T is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.]
9. Use the Brouwer fixed point theorem to prove that there is a complex number z with $|z| \leq 1$ such that $z^4 - z^3 + 8z^2 + 11z + 1 = 0$.

10. Let $C[0, 1]$ be the metric space consisting of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$, with the distance $d(f, g)$ between two functions f and g defined to be the supremum of $|f(x) - g(x)|$ over all $x \in [0, 1]$.

(i) Explain why this supremum is in fact a maximum.

(ii) Let X consist of all functions f in $C[0, 1]$ that take values in $[0, 1]$. Prove that X is not a sequentially compact subset of $C[0, 1]$.

(iii) This shows that X is not compact. Prove the same result by exhibiting an open cover of X that has no finite subcover.

11. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has the intermediate value property that if $a, b \in \mathbb{R}$ and $f(a) < c < f(b)$, then $f(y) = c$ for some y between a and b .

(i) Must f be continuous? (Justify your answer.)

(ii) If additionally f has the property that $f^{-1}(a)$ is closed for every $a \in \mathbb{Q}$, show that f is continuous.

12. Let q be an integer ≥ 2 and n an integer ≥ 1 . Denote by \mathcal{Q} the set of all unordered q -tuples of points in \mathbb{R}^n . Thus $\mathcal{Q} = \{\{x_1, x_2, \dots, x_q\} : x_1, x_2, \dots, x_q \text{ are not necessarily distinct points in } \mathbb{R}^n\}$. Define a function $\mathcal{G} : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathbb{R}$ by

$$\mathcal{G}(\{x_1, x_2, \dots, x_q\}, \{y_1, y_2, \dots, y_q\}) = \inf \left\{ \left(\sum_{j=1}^q |y_j - x_{\sigma(j)}|^2 \right)^{1/2} : \sigma \text{ is a permutation of } 1, 2, \dots, q \right\}.$$

(i) Show that \mathcal{G} is a metric on \mathcal{Q} .

(ii) Assume $n = 1$. Note that in this case, we may, for any point $x = \{x_1, x_2, \dots, x_q\} \in \mathcal{Q}$, choose the labelling so that $x_1 \leq x_2 \leq \dots \leq x_q$. Assuming we have done this for all points

in \mathcal{Q} , is it true that $\mathcal{G}(\{x_1, x_2, \dots, x_q\}, \{y_1, y_2, \dots, y_q\}) = \left(\sum_{j=1}^q (x_j - y_j)^2 \right)^{1/2}$?