

## TOPICS IN ANALYSIS (Lent 2014): Example Sheet 3.

Comments, corrections are welcome at any time.

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1. For each positive integer  $n$  and  $k \in \{1, 2, \dots, n\}$ , let the non-negative numbers  $A_k^{(n)}$  and the ‘nodes’  $x_k^{(n)} \in [a, b]$  be given such that for each polynomial  $P$ , the error

$$\varepsilon_n(P) = \left| \int_a^b P(x) dx - \sum_{k=1}^n A_k^{(n)} P(x_k^{(n)}) \right|$$

in approximating  $\int_a^b P(x) dx$  by  $\sum_{k=1}^n A_k^{(n)} P(x_k^{(n)})$  tends to zero as  $n \rightarrow \infty$ . Prove that  $\varepsilon_n(f) \rightarrow 0$  for each continuous function  $f$  on  $[a, b]$ .

2. Let  $n$  be a positive integer. We proved in lectures that there are  $n$  distinct points  $\alpha_1, \alpha_2, \dots, \alpha_n \in [-1, 1]$  and  $n$  real numbers  $A_1, A_2, \dots, A_n$  such that the formula

$$\int_{-1}^1 p(x) dx = \sum_{j=1}^n A_j p(\alpha_j)$$

is valid for every polynomial  $p$  of degree  $\leq 2n - 1$ . (The  $\alpha_j$  are in fact the zeros of the  $n$ th Legendre polynomial over  $[-1, 1]$ ). Is it possible to find such  $n$  distinct points in  $[-1, 1]$  and numbers  $A_1, A_2, \dots, A_n$  such that the above formula is valid for every polynomial  $p$  of degree  $\leq 2n$ ?

3. Let  $T_j$  be the  $j$ th Chebyshev polynomial. Suppose  $\gamma_j$  is a sequence of non-negative numbers with  $\sum_{j=1}^{\infty} \gamma_j < \infty$ . Prove that  $\sum_{j=1}^{\infty} \gamma_j T_{3^j}$  defines a continuous function  $f$  on  $[-1, 1]$  with the following property. For each  $n$ , there exist points  $-1 \leq x_0 < x_1 < \dots < x_{3^{n+1}} \leq 1$  such that, writing  $P_n$  for the partial sum  $\sum_{j=1}^n \gamma_j T_{3^j}$ ,

$$f(x_k) - P_n(x_k) = (-1)^k \sum_{j=n+1}^{\infty} \gamma_j$$

holds for each  $k = 0, 1, \dots, 3^{n+1}$ .

4. For each  $f \in C([-1, 1])$ , let  $E_n(f)$  be the distance from  $f$  to the subspace  $\mathcal{P}_n$  of polynomials of degree at most  $n$ . That is,  $E_n(f) = \inf_{p \in \mathcal{P}_n} \sup_{x \in [-1, 1]} |f(x) - p(x)|$ . We know by the Weierstrass approximation theorem that  $E_n(f) \rightarrow 0$  for each  $f \in C([-1, 1])$ . Using the result of Question 3, construct a function  $f \in C([-1, 1])$  to show that the convergence  $E_n(f) \rightarrow 0$  can be arbitrarily slow in the following sense. For any given decreasing sequence of non-negative numbers  $\delta_n$  converging to zero, there exists  $f \in C([-1, 1])$  such that  $E_n(f) \geq \delta_n$  for all  $n = 1, 2, \dots$ .

**5.** For  $n, r \in \mathbb{Z}$  and  $n \geq 1$ , define  $\Delta_{n,r} : [-1, 1] \rightarrow \mathbb{R}$  by  $\Delta_{n,r}(x) = \max\{0, 1 - n|x - rn^{-1}|\}$ . Sketch  $\Delta_{n,r}$ .

Now consider  $f : [-1, 1] \rightarrow \mathbb{R}$ . Show that  $f_n(x) = \sum_{m=-n}^n f(m/n) \Delta_{n,m}(x)$  is a piecewise linear function with  $f_n(r/n) = f(r/n)$ .

Show that, if  $f$  is continuous, then  $\|f_n - f\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ .

**6.** Use the result of Question 5 to prove that there exists a sequence of functions  $\phi_n \in C([-1, 1])$ ,  $n = 0, 1, 2, \dots$ , such that for every  $f \in C([-1, 1])$ , there exists a unique series  $\sum_{n=0}^\infty a_n \phi_n$  which converges uniformly to  $f$ .

**7.** For each  $n = 1, 2, \dots$ , let  $f_n : [0, 1] \rightarrow \mathbb{R}$  be functions such that  $f_n$  converge uniformly to a function  $f$ . Suppose also that  $f$  is bounded. Prove that for any positive integer  $m$  the functions  $g_n(t) = f_n(t)^m$  converge uniformly on  $[0, 1]$  to  $g(t) = f(t)^m$ .

**8.** Let  $B_r(z)$  denote the open ball about  $z$  with radius  $r$  in the complex plane and let  $U = B_2(1) \setminus \overline{B_1(0)}$ . Suppose that  $f$  is holomorphic in  $U$ .

(i) Prove that there exists a sequence of polynomials which converges to  $f$  uniformly on compact subsets of  $U$ .

(ii) Must there be a sequence of polynomials which converges to  $f$  uniformly on  $U$ ?

(iii) If additionally we assume that  $f$  is holomorphic in some open set containing the closure of  $U$ , must there be a sequence of polynomials which converges to  $f$  uniformly on  $U$ ?

**9.** Construct a sequence of polynomials which converges uniformly to  $1/z$  on the semicircle  $\{z : |z| = 1, \Re(z) \geq 0\}$ .

**10.** Let  $U$  be a bounded open subset of the complex plane  $\mathbb{C}$  such that  $\mathbb{C} \setminus U$  is connected. Prove that  $f : U \rightarrow \mathbb{C}$  is holomorphic if and only if whenever  $K$  is a compact subset of  $U$  and  $\epsilon > 0$  we can find a polynomial  $P$  such that

$$|f(z) - P(z)| < \epsilon$$

for all  $z \in K$ . (This gives yet another of several equivalent definitions of holomorphic functions.)

**11.** Does there exist a sequence of complex polynomials  $p_n$  such that  $p_n(0) = 1$  for every  $n = 1, 2, \dots$  and  $p_n(z) \rightarrow 0$  for each  $z \in \mathbb{C} \setminus \{0\}$ ?

**12.** Let  $A = \{z \in \mathbb{C} : 1/2 \leq |z| \leq 1\}$ , and let  $f : A \rightarrow \mathbb{C}$  be continuous in  $A$  and holomorphic in the interior of  $A$ . If there exists a sequence of complex polynomials converging uniformly on  $A$  to  $f$ , prove that there exists a continuous function  $g : \{z : |z| \leq 1\} \rightarrow \mathbb{C}$  such that  $g$  is holomorphic in  $\{z : |z| < 1\}$  and  $g(z) = f(z)$  for every  $z \in A$ . [Hint: if  $p_n$  are polynomials converging uniformly to  $f$  on  $A$ , apply the maximum modulus principle to  $p_n - p_m$  over a suitable domain.]