Topics in Analysis, Examples Sheet 2, 2013

$\|\cdot\|$ denotes the uniform norm on $C([0,1])$.

1. Let $f:[0,1]\to\mathbf{R}$ be a continuous function with the property that $\int_0^1 f(x)x^n dx = 0$ for $n = 0,1,2,\dots$. Prove that f is identically zero.

2. Suppose that $f:[0,1]\to\mathbf{R}$ is uniformly approximated by polynomials, but is not itself a polynomial. Show that the degrees of these approximating polynomials tend to infinity.

3. Suppose that $f:[0,1]\to\mathbf{R}$ is continuous. Let $n\geqslant 1$ be an integer. Show that there is a unique polynomial p of degree at most n for which $\|f-p\|_2 = (\int_0^1 |f(x)-p(x)|^2 dx)^{1/2}$ is minimal.

4. A metric space X is called *separable* if it has a countable dense subset S (thus every ball $B_\varepsilon(x)$ in X contains some point of S). Show that the space $C([0,1])$ of continuous functions on $[0,1]$ is separable.

5. Prove the following extension of the Weierstrass approximation theorem. Given $\varepsilon>0$, a continuous function $f:[0,1]\to\mathbf{R}$, and points $x_1,\dots,x_m\in[0,1]$, there is a polynomial p such that $\|f-p\|\leqslant\varepsilon$ and $f(x_i)=p(x_i)$ for all i.

6. Let $f:[0,\pi]\to\mathbf{R}$ be a continuous function with $f(0)=f(\pi)$. Show that f can be uniformly approximated by trigonometric polynomials of the form $\sum_{n\in\mathbf{Z}} a_n\cos nx$. Now suppose that $f:[0,2\pi]\to\mathbf{R}$ is a continuous function with $f(0)=f(2\pi)$. Show that f can be uniformly approximated by trigonometric polynomials of the form $\sum_n a_n\cos nx+\sum_n b_n\sin nx$.

7. Suppose that $f:[0,1]\to\mathbf{R}$ satisfies the Lipschitz property $|f(x)-f(y)|\leqslant|x-y|$. Show that there is a polynomial p of degree at most $10/\varepsilon^3$ such that $\|f-p\|\leqslant\varepsilon$.

8. Prove that every continuous function $f:[0,1]\times[0,1]\to\mathbf{R}$ can be uniformly approximated by polynomials in two variables.

9. Suppose that p is a polynomial of degree n with complex coefficients. If $|p(x)|\leqslant 1$ for all *real* x such that $|x|\leqslant 1$, show that every coefficient of p has magnitude at most $(10n)^n$. If $|p(z)|\leqslant 1$ for all *complex* z such that $|z|\leqslant 1$, show that every coefficient of p has magnitude at most 1.

10. The aim of this exercise is to give what I believe to be some kind of approximation to Weierstrass's original proof of the approximation theorem. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous, and vanishes outside of some bounded interval.

(1) Define $K_n(x) = \sigma_n e^{-nx^2}$, where σ_n is chosen so that $\int_{-1}^{1} K_n(x)dx = 1$. Show that $\sigma_n \leqslant C\sqrt{n}$ for some absolute constant C, and give a sketch of the functions K_n.

(2) Let $\delta > 0$. Show that $\int_{|t|\geqslant\delta} K_n(t)dt \to 0$ as $n \to \infty$.

(3) Consider the convolution $f * K_n(x) = \int_{\mathbf{R}} f(x-t)K_n(t)dt$. Show that $f * K_n \to f$ uniformly on any bounded interval. (Hint: look again at the argument given in lectures, and use the previous part of the question.)

(4) Show that if K is a polynomial, then so is $f * K$.

(5) Show that for each fixed n and bounded interval I, K_n may be uniformly approximated by polynomials on I.

(6) Deduce the Weierstrass approximation theorem.

11. The aim of this exercise is to prove the following theorem: if $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a complex polynomial of degree n with $|a_i| = 1$ for all i then the multiplicity of 1 as a root of p is no more than $C\sqrt{n}$ for some constant C.

(1) Consider first the polynomial $f(z) = \frac{1}{2}T_0(z) + T_1(z) + \cdots + T_k(z)$, where the T_i are the Chebyshev polynomials. Prove that $|f(z)| \leqslant (2(1-z))^{-1/2}$ for $z \in [-1, 1)$, and that $f(1) = k + \frac{1}{2}$.

(2) Now define $g(z) = (f(1 - \frac{2z}{n})^4$. Show that $|g(1)| + \cdots + |g(n)| < g(0)$ if $k > 10\sqrt{n}$.

(3) Show that if p had a zero of order $4k + 1$ at 1 then we would have $a_0 g(0) + \cdots + a_n g(n) = 0$, and hence deduce the theorem.

12. Suppose we are given four distinct points A, B, C, D on the unit circle (thought of as a subset of $\mathbf{R}^2$), in that order. Suppose that γ is a continuous path from A to C, and that γ' is a continuous path from B to D, and that both of these paths lie inside the closed unit disc. Show that they meet.

13. Suppose that $f : [0, 1] \to \mathbf{R}$ is continuous. Let $n \geqslant 1$ be an integer. Show that there is a unique polynomial p of degree at most n for which $\|f - p\|_1 = \int_0^1 |f(x) - p(x)|dx$ is minimal. (Hints: suppose that F is a continuous function with finitely many zeros which is n-unimprovable in the sense that $\|F + p\|_1 \geqslant \|F\|_1$ for all p of degree at most n. Show that there cannot be a polynomial q of degree at most n whose sign always coincides with that of F. Conclude that F has at least $n + 1$ crossing zeros in $(0, 1)$, and then finish as in lectures).

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