Topics in Analysis, Examples Sheet 1, 2013

1. For each of the following sets X (each of which is endowed with the Euclidean metric) give an example of a continuous map $f : X \to X$ without a fixed point: (i) X is the open unit disc; (ii) X is a closed square of sidelength 1, from which an open square of sidelength $\frac{1}{2}$ with the same centre has been removed; (iii) $X = \mathbf{Q} \cap [0, 1]$.

2. Suppose that A and B are disjoint, closed, bounded subsets of $\mathbf{R}^2$. Show that there is a $\delta > 0$ so that $|a - b| \geqslant \delta$ for all $a \in A$ and $b \in B$. Is the same true if just one of A and B is bounded? What if neither is bounded?

3. Let $f : [0, 1] \to \mathbf{R}$ be a function. Suppose that f has the intermediate value property. Is f continuous?

4. Prove carefully that homotopy equivalence of closed paths in $\mathbf{R}^2$ is an equivalence relation.

5. Let T be an n-dimensional simplex. Come up with a sensible definition of what it means for T to be *regular* and prove that there is a regular n-simplex for each n. Prove that T admits a triangulation into elementary n-simplices of diameter at most ϵ.

6. Let $x \in \mathbf{Q}$, define the 2-adic valuation $|x|_2$ by setting $|x|_2 = 2^{-n}$ if $x = 2^n \frac{p}{q}$ with p and q odd, for some $n \in \mathbf{Z}$. Show that if we define $d(x, y) = |x - y|_2$ then this defines a metric on $\mathbf{Q}$. Is $\mathbf{Q}$ complete with respect to this metric?

7. Let X be a sequentially compact metric space, and suppose that $f : X \to X$ is a continuous map. Show that $f(X)$ is closed. We say that f is an *isometry* if $d(f(x), f(x')) = d(x, x')$ for all $x, x' \in X$. Show that isometries are automatically continuous. Show furthermore that f is surjective. [*Hint:* pick $x \notin f(X)$, and consider the sequence of iterates $x, f(x), f^2(x), f^3(x), \dots$]

8. Let X be a complete metric space, and suppose that $f : X \to X'$ is a homeomorphism, that is to say a continuous map with a continuous inverse. Is X' necessarily complete?

9. Use the Brouwer fixed point theorem to show that there is a complex number z with $|z| \leqslant 1$ so that $z^4 - z^3 + 8z^2 + 11z + 1 = 0$.

10. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous, and suppose that $f(x)$ differs from x by at most 10. Show that f has a fixed point.

11. Let A be a square matrix with positive entries. Using the Brouwer fixed point theorem, prove that A has an eigenvector with positive entries.

12. Let X be the metric space consisting of all continuous functions $f : [0,1] \to \mathbf{R}$, with metric $d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$. Show that this is a metric space and explain, using results from IB, why X is complete. Consider the subset $Y \subset X$ consisting of all functions f with $|f(y)| \leqslant 1$. Is Y sequentially compact?

For enthusiasts.

13. (1) Show that we may extend the valuation $|\cdot|_2$ on $\mathbf{Q}$ to a valuation on $\mathbf{R}$, still satisfying the basic properties that $|x|_2 = 0$ if and only if $x = 0$, that $|xy|_2 = |x|_2|y|_2$, and that $|x+y|_2 \leqslant \max(|x|_2, |y|_2)$.

(2) Colour the unit square as follows. Colour (x,y) red if $|x|_2 < 1$ and $|y|_2 < 1$, blue if $|x|_2 \geqslant 1$ and $|x|_2 \geqslant |y|_2$, and green if $|y|_2 \geqslant 1$ and $|y|_2 > |x|_2$. Show that any no line contains points of all three colours.

(3) Now consider a triangulation of the unit square. By slightly adapting Sperner's lemma, show that there is a triangle with vertices of all three colours.

(4) Show that if a multicoloured triangle has area A then $|A|_2 > 1$.

(5) Conclude that it is impossible to partition the unit square into n triangles, each with the same area, if n is odd.

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