Topics in Analysis: Example Sheet 3

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(1) For each positive integer n and $k \in \{1, 2, \ldots, n\}$, let the non-negative numbers $A_k^{(n)}$ and the "nodes" $x_k^{(n)} \in [a, b]$ be given such that for each polynomial P, the error

$$\epsilon_n(P) \equiv \left| \int_a^b P(x)\, dx - \sum_{k=1}^{n} A_k^{(n)} P(x_k^{(n)}) \right|$$

in approximating $\int_a^b P(x)\, dx$ by $\sum_{k=1}^{n} A_k^{(n)} P(x_k^{(n)})$ tends to zero as $n \to \infty$. Prove that $\epsilon_n(f) \to 0$ for each continuous function f on $[a, b]$.

(2) Let n be a positive integer. We proved in lectures that there are n distinct points $\alpha_1, \alpha_2, \ldots, \alpha_n \in [-1, 1]$ and n real numbers $A_1, A_2, \ldots, A_n$ such that the formula

$$\int_{-1}^{1} p(x)\, dx = \sum_{j=1}^{n} A_j p(\alpha_j)$$

is valid for every polynomial p of degree $\leq 2n - 1$. (The α_j are in fact the zeros of the nth Legendre polynomial over $[-1, 1]$). Is it possible to find such n distinct points in $[-1, 1]$ and numbers $A_1, A_2, \ldots, A_n$ such that the above formula is valid for every polynomial p of degree $\leq 2n$?

(3) Let T_j be the jth Chebychev polynomial. Suppose γ_j is a sequence of non-negative numbers with $\sum_{j=1}^{\infty} \gamma_j < \infty$. Prove that $\sum_{j=1}^{\infty} \gamma_j T_{3^j}$ defines a continuous function f on $[-1, 1]$ with the property that for each n, there exist points $-1 \leq x_0 < x_1 < \ldots < x_{3^{n+1}} \leq 1$ such that, writing P_n for the partial sum $\sum_{j=1}^{n} \gamma_j T_{3^j}$,

$$f(x_k) - P_n(x_k) = (-1)^k \sum_{j=n+1}^{\infty} \gamma_j$$

for each $k = 0, 1, \ldots, 3^{n+1}$.

(4) For each $f \in C([-1, 1])$, let $E_n(f)$ be the distance from f to the subspace $\mathcal{P}_n$ of polynomials of degree at most n. That is, $E_n(f) = \inf_{p \in \mathcal{P}_n} \sup_{x \in [-1,1]} |f(x) - p(x)|$. We know by the Weierstrass approximation theorem that $E_n(f) \to 0$ for each $f \in C([-1, 1])$. Using the result of problem (2), construct a function $f \in C([-1, 1])$ to show that the convergence $E_n(f) \to 0$ can be arbitrarily slow in the following sense. For any given decreasing sequence of non-negative numbers δ_n converging to zero, there exists $f \in C([-1, 1])$ such that $E_n(f) \geq \delta_n$ for all $n = 1, 2, \ldots$.

(5) Let $f \in C([0, 1])$ and let $\{q_0, q_1, q_2, q_3, \ldots\}$ be a dense set of distinct points in $[0, 1]$ with $q_0 = 0$ and $q_1 = 1$. For $n = 1, 2, \ldots$, let f_n be the piecewise linear function with $f_n(q_j) = f(q_j)$ for $j = 0, 1, \ldots, n$. Prove that $f_n \to f$ uniformly on $[0, 1]$.

(6) Use the result of problem (5) to prove that there exists a sequence of functions $\varphi_n \in C([0, 1])$, $n = 0, 1, 2, \ldots$, such that for every $f \in C([0, 1])$, there exists a unique series $\sum_{n=0}^{\infty} a_n \varphi_n$ which

converges uniformly to f.

(7) Let $f : [-1,1] \to \mathbf{R}$ be a function and n an integer ≥ 0. Show that there can be at most one polynomial P of degree $\leq n$ such that $|f(x) - P(x)| \leq M|x|^{n+1}$ for some constant $M > 0$ and all $x \in [-1,1]$.

(8) Let Ω be a subset of the complex plane $\mathbf{C}$ and for each $n = 1, 2, 3, \ldots$, let $f_n : \Omega \to \mathbf{C}$ be functions such that f_n converge uniformly on Ω to a bounded function $f : \Omega \to \mathbf{C}$. Prove that for any fixed positive integer m, $f_n^m \to f^m$ uniformly.

(9) Let $B_r(z)$ denote the open ball in the complex plane with radius r and centre z and let $\Omega = B_2(1) \setminus \overline{B_1(0)}$. Suppose that f is analytic in Ω.
(a) Prove that there exists a sequence of polynomials which converges to f uniformly on compact subsets of Ω.
(b) Must there be a sequence of polynomials which converges to f uniformly on Ω?
(c) If additionally we assume that f is analytic in some open set containing the closure of Ω, must there be a sequence of polynomials which converges to f uniformly on Ω?

(10) Construct a sequence of polynomials which converges uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \mathrm{Re}(z) \geq 0\}$.

(11) Let Ω be a bounded open subset of the complex plane $\mathbf{C}$ such that $\mathbf{C} \setminus \Omega$ is connected. If $f : \Omega \to \mathbf{C}$ is analytic, prove that there exists a sequence of polynomials converging uniformly to f on each compact subset of Ω.

(12) Does there exist a sequence of complex polynomials p_n such that $p_n(0) = 1$ for every $n = 1, 2, 3, \ldots$ and $p_n(z) \to 0$ for each $z \in \mathbf{C} \setminus \{0\}$?

(13) Let $A = \{z \in \mathbf{C} : 1/2 \leq |z| \leq 1\}$, and let $f : A \to \mathbf{C}$ be continuous in A and analytic in the interior of A. If there exists a sequence of complex polynomials converging uniformly on A to f, prove that there exists a continuous function $g : \{z : |z| \leq 1\} \to \mathbf{C}$ such that g is analytic in $\{z : |z| < 1\}$ and $g(z) = f(z)$ for every $z \in A$. (Hint: if p_n are polynomials converging uniformly to f on A, apply the maximum modulus principle to $p_n - p_m$ over a suitable domain.)

(14) (Optional) Formulate and prove a generalization of the result in (13) to a setting where A is an arbitrary compact subset of the complex plane. Contrast your result with Runge's theorem proved in lecture.