**Topics in Analysis: Example Sheet 4**

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(1) Determine the continued fraction expansions of 71/49 and $\sqrt{3}$. Deduce that $\sqrt{3}$ is irrational.

(2) Show that the simple continued fraction representing an irrational number $\alpha$ is uniquely determined by $\alpha$.

(3) Let $a$, $b$ be positive integers. Let

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \dots}}}}.$$

(a) Show that $x$ solves $ax^2 + abx - b = 0$.

(b) Assume $a = b = 1$. Show that, in this case, $x = \frac{-1+\sqrt{5}}{2}$ and that if $\frac{p_n}{q_n}$ is the $n$th convergent of the continued fraction, then $p_n = F_n$, $q_n = F_{n+1}$, where $F_0, F_1, F_2, \dots$ is the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

(c) Show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^{n+1}$.

(4) Let $p$ be a positive integer and $\alpha, \beta$ be real numbers such that $\alpha + \beta = \alpha\beta = -p$. Find the simple continued fraction representations of $|\alpha|$ and $|\beta|$ in terms of $p$.

(5) Determine the rational number with denominator $\leq 10$ that best approximates 71/49.

(6) Recall that a topological space $(X, \tau)$ is said to be a $T_1$ space if for each pair $x, y$ of distinct points of $X$, there exists an open set $U$ such that $x \in U$ and $y \notin U$. Prove that the following are equivalent:

(a) $(X, \tau)$ is a $T_1$ space.

(b) $\{x\}$ is a closed subset of $X$ for every point $x \in X$.

(c) If $A$ is any subset of $X$, then $A = \cap_{U \in \tau,\, A \subseteq U} U$.

(7) Let $f : [1, \infty) \to \mathbf{R}$ be a continuous function and suppose that for each $x \in [1, \infty)$, $f(nx) \to 0$ as $n \to \infty$, $n \in \mathbf{N}$. Prove that $\lim_{x \to \infty} f(x) = 0$. (Hint: For $\epsilon > 0$, consider the sets $Q_k = \{x \in [1, \infty) : |f(nx)| < \epsilon \quad \forall n \geq k\}$.)

(8) Is the set of irrationals, as a subset of $\mathbf{R}$, of second category?

(9) Let $A_j$ be a sequence of subsets of $[0, 1]$ such that for each $N \geq 1$, $\cup_{j=N}^{\infty} A_j$ is open and dense in $[0, 1]$. Prove that the set $S$ of points $x \in [0, 1]$ such that $x \in A_j$ for infinitely many $j$ is dense. Must $S$ be open? Must it be true that $\cap_{j=1}^{\infty} A_j \neq \emptyset$?

(10) If $G$ is an open dense subset of $\mathbf{R}$, and $\mathbf{Q}$ is the set of rationals, show that $G \setminus \mathbf{Q}$ must be dense in $\mathbf{R}$. If we only assume $G$ is uncountable and dense in $\mathbf{R}$, does it still follow that $G \setminus \mathbf{Q}$ is dense in $\mathbf{R}$?

(11) (Principle of Uniform Boundedness.) Let $(X, d)$ be a complete metric space and let $\mathcal{F}$ be a family of real valued continuous functions on $X$. Suppose for each $x \in X$, the set $\{f(x) : f \in \mathcal{F}\}$ is bounded. (i.e. for each $x \in X$, there exists a number $M_x \geq 0$, possibly depending on $x$, such that $|f(x)| \leq M_x$ for each $f \in \mathcal{F}$.) Prove that there exists a ball $B_r(x_0) \subset X$ and a number $M \geq 0$ such that

$$|f(x)| \leq M$$

for each $x \in B_r(x_0)$ and $f \in \mathcal{F}$.

(12) Let $f : \mathbf{C} \to \mathbf{C}$ be a continuous function. If there is a positive integer $n$ and a non-zero complex number $c$ such that

$$\lim_{z \to \infty} z^{-n} f(z) = c,$$

prove that $f(z_0) = 0$ for some $z_0 \in \mathbf{C}$.

(13) Let $\gamma : [0, 1] \to \mathbf{C} \setminus \{0\}$ be a continuous map such that $\gamma(0) = \gamma(1)$ and suppose that for each $t \in [0, 1]$, $\gamma(t)$ is not equal to a negative real number. By considering $\gamma(t) + c$ for suitable values of $c \in [0, \infty)$, or otherwise, show that the winding number $w(\gamma; 0) = 0$.

(14) Let $g : \mathbf{S}^1 \to \mathbf{S}^1$ be a continuous map, where $\mathbf{S}^1 = \{z \in \mathbf{C} : |z| = 1\}$. If there is a continuous extension of $g$ to the closed unit disk $D = \{z \in \mathbf{C} : |z| \leq 1\}$ (i.e. if there is a continuous map $G : D \to \mathbf{S}^1$ such that $G(z) = g(z)$ for each $z \in \mathbf{S}^1$), prove that

(a) $g(z) = z$ for some $z \in \mathbf{S}^1$.

(b) $g(z) = -z$ for some $z \in \mathbf{S}^1$.

(15) (Optional.) (a) Let $f : [0, 1] \to \mathbf{R}$ be a non-negative, bounded Riemann integrable function with $\int_0^1 f = 0$. If $E = \{x \in [0, 1] : f(x) \neq 0\}$, prove that $E$ is a set of Lebesgue measure zero.

(b) Let $f : [0, 1] \to \mathbf{R}$ be a bounded Riemann integrable function. If

$$\int_0^1 f(x) x^n \, dx = 0 \quad \text{for} \ \ n = 0, 1, 2, \ldots,$$

prove that $f(x) = 0$ except on a set of Lebesgue measure zero.