Topics in Analysis: Example Sheet 1

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(1) Let X be a subset of $\mathbf{R}^n$. If every continuous function $f : X \to \mathbf{R}$ is bounded, prove that X is compact.

(2) Consider the metric space $(\mathbf{Q}, d)$ where $\mathbf{Q}$ is the set of rational numbers and d is the usual Euclidean metric. Let P be the set of all $p \in \mathbf{Q}$ such that $2 < p^2 < 3$. Prove that P is closed and bounded in $\mathbf{Q}$, but not compact. Give an open cover of P which has no finite subcover.

(3) Find a metric d on $X = (0, 1]$ such that (X, d) is a complete metric space, and such that a subset of X is open with respect to d if and only if it is open with respect to the usual Euclidean metric.

(4) Prove that the following statements are equivalent. (Note that this is the 1-dimensional version of the 2-dim theorem proved in lecture).

(a) If $f : [-1, 1] \to [-1, 1]$ is continuous, then there exists a number $y \in [-1, 1]$ such that $f(y) = y$.

(b) There is no continuous function $g : [-1, 1] \to \{-1, 1\}$ with $g(-1) = -1$ and $g(1) = 1$.

(c) If A and B are closed subsets of $\mathbf{R}$ such that $-1 \in A$, $1 \in B$ and $[-1, 1] \subset A \cup B$, then $A \cap B \neq \emptyset$.

(d) If $h : [-1, 1] \to \mathbf{R}$ is continuous and c is a number between $f(-1)$ and $f(1)$, then there exists a number $x \in [-1, 1]$ such that $f(x) = c$.

(5) Prove the statement (b) of problem (4) above by completing the following. (Note that this is the 1-dim version of the proof given in lecture for the 2-dim case, which was based on Green's theorem).

(i) Show that if there is a continuous function $g : [-1, 1] \to \{-1, 1\}$ with $g(-1) = -1$ and $g(1) = 1$, then there is also a smooth function $\widetilde{g} : \mathbf{R} \to \mathbf{R} \setminus \{0\}$ such that $\widetilde{g}(x) = x$ for all x with $|x| \geq 1 - \sigma$ for an appropriate choice of $\sigma \in (0, 1/2)$.

(ii) Apply the fundamental theorem of calculus $G(1) - G(-1) = \int_{-1}^{1} \frac{dG}{dt} dt$, with justification, to $G(t) = F(\widetilde{g}(t))$ for an appropriate function $F : \mathbf{R} \setminus \{0\} \to \mathbf{R}$ to a get a contradiction.

(6) Let $f : B \to B$ be a continuous map from the *open* disc $B = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}$ into itself. Does it follow that f has a fixed point? Does your answer change if f is a continuous bijection?

(7) Suppose that A is a 3×3 matrix with positive entries. By considering a suitable map from the triangle $T = \{\mathbf{x} \in \mathbf{R}^3 : x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}$ into itself, prove that A has an eigenvector with positive entries.

(8) Suppose (X, d) is a compact metric space and $f : X \to X$ is a continuous function such that $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove that f is a surjection. (Hint: if not show that there is a point $y \in X \setminus f(X)$ and $r > 0$ such that $B_r(y) \subset X \setminus f(X)$, and consider the sequence

$y, f(y), f(f(y)), \ldots.)$

(9) Recall that a function $f : X \to Y$ between metric spaces is uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x, y \in X$, $d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$. If (X, d_X) is compact, prove that every continuous function $f : X \to Y$ is uniformly continuous. Give in fact two proofs; a direct proof using the definition of compactness involving open covers, and another proof by contradiction using sequential compactness.

(10) Suppose $f : \mathbf{R} \to \mathbf{R}$ has the intermediate value property that if $a, b \in \mathbf{R}$, $f(a) < c < f(b)$, then $f(y) = c$ for some y between a and b.

(i) Show that f need not be continuous.

(ii) If additionally f has the property that $f^{-1}(\{a\})$ is closed for every a in a dense subset of $\mathbf{R}$, show that f is continuous.

(11) Let D be the closed unit disc in $\mathbf{R}^2$, $x_0 \in D$ and suppose that $f : D \to D$ is a continuous function such that $f(x_0) \neq x_0$.

(i) Show that there is $r > 0$ such that $f(x) \neq x$ for all $x \in B_r(x_0)$.

(ii) For $x \in B_r(x_0)$, define $g(x)$ to be the point where ∂D meets the ray obtained by extending the line segment connecting $f(x)$ to x in the direction from $f(x)$ to x. Prove that $g : B_r(x_0) \to \partial D$ is continuous at x_0.

(Hint: note that any point on the ray in question can be written as $f(x) + \tau(x - f(x))$ for some $\tau \geq 0$, and for each $x \in D$, the requirement that this point belongs to ∂D determines uniquely a non-negative $\tau = \tau(x)$.)

(12) Let q be an integer ≥ 2 and n an integer ≥ 1. Denote by $\mathcal{Q}$ the set of all unordered q-tuples of points in $\mathbf{R}^n$. Thus $\mathcal{Q} = \{\{x_1, x_2, \ldots, x_q\} : x_1, x_2, \ldots, x_q \text{ are not necessarily distinct points } \in \mathbf{R}^n\}$. Define a function $\mathcal{G} : \mathcal{Q} \times \mathcal{Q} \to \mathbf{R}$ by

$$\mathcal{G}(\{x_1, x_2, \ldots, x_q\}, \{y_1, y_2, \ldots, y_q\}) = \inf \left\{ \left(\sum_{j=1}^{q} |x_j - x_{\sigma(j)}|^2 \right)^{1/2} : \sigma \text{ is a permutation of } 1, 2, \ldots, q \right\}.$$

(a) Show that $\mathcal{G}$ is a metric on $\mathcal{Q}$.

(b) Assume $n = 1$. Note that in this case, we may, for any point $x = \{x_1, x_2, \ldots, x_q\} \in \mathcal{Q}$, choose the labeling so that $x_1 \leq x_2 \leq \ldots \leq x_q$. Assuming we have done this for all points in $\mathcal{Q}$, is it true that $\mathcal{G}(\{x_1, x_2, \ldots, x_q\}, \{y_1, y_2, \ldots, y_q\}) = \left(\sum_{j=1}^{q} (x_j - y_j)^2\right)^{1/2}$?

(13) (Optional). If $f_n : [0,1] \to \mathbf{R}$, $f_n(0) = 0$ and $|f_n(x) - f_n(y)| \leq |x - y| + \frac{1}{n}$ for each $x, y \in [0,1]$ and $n = 1, 2, \ldots$, prove that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ converges uniformly on $[0,1]$ to a continuous function.